A DISCUSSION OF SOME DETERMINISTIC AND PROBABILISTIC INVENTORY MODELS

A THESIS

SUBMITTED FOR

THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

By

PRAMOD NARAIN MISHRA

BUNDELKHAND UNIVERSITY, JHANSI (INDIA)

1996

CERTIFICATE

This is to certify that the work embodied in the thesis, "A DISCUSSION OF SOME DETERMINISTIC AND PROBABILISTIC INVENTORY MODELS" by Pramod Narain Mishra, for the award of the degree of Doctor of Philosophy is a record of bonafide research work carried out by him under my supervision and guidance and has not been submitted elsewhere for a degree/diploma in any form.

It is further certified that he has worked with me for the period required under clause seven of Bundelkhand university ordinance.

H. Kishan

Dr. Hari Kishan
Department of Mathematics
D.N. College, Meerut (U.P.) India
Formerly
Lecturer
Department of Mathematics
Atarra Post Graduate College
Atarra, Banda (U.P.) India

ACKNOWLEDGEMENTS

It is my great privilege to express my deep sense of gratitude to my esteemed supervisor Dr. Hari Kishan, Department of Mathematics D.N. college, Meerut (Formerly Lecturer, Department of Mathematics Atarra college Atarra, Banda) for his suggestions, constructive criticisms and invaluable guidance, words fail me to express my feelings for the affection and whole hearted support, I have received from him.

I take this opportunity to thank Dr. B.L. Gaur, Principal Atarra College Atarra, Banda and Dr. B.N. Dwedi, Head department of Mathematics, Atarra College Atarra, Banda for their sympathetic help and cooperation.

My sincere thanks are due to Sri A.K. Singh, Principal Narmada College of Science and commerce, Bharuch for his constant encouragement and moral support.

It is pleasure for me to put on record my friends Dr. G.C. Sharma, Dr. Dinesh Kumar, Mr. Mukul Awasthi, Mr. Mridul Gupta and other colleagues of the college for their kind cooperation, I got during the entire course of my work.

I am also grateful to Mrs. Karuna for enlightening my confidence from time to time during the preparation of this thesis. I also can not forget the love showered by the littles Chinu, Sunny and Rinki, I wish a happy and long life for them.

I humbly dedicate this work to my parents Mrs. and Mr. M.L.Mishra who have been a source of inspiration for me and not only sustained me through many difficult times with tender care and affection but also worked untiringly to provide me an opportunity for higher education.

For the serious and time consuming work of protracted nature like the research work needs, peaceful and undisturbed environment. My wife Rekha, has not only provided the very conducive atmosphere for completing the study but has also with stood all the strains and stresses of managing and looking after me during my long business in this work, I owe a feeling of deep gratitute towards her.

I am also indebted to my son Master Saurabh, who, I feel has been deprived of my love and care during this period.

I am also thankful to Shri. Rajiv Jain for the pains, he has taken in typing and composing of the thesis carefully.

Pramod Narain Mishra
Department of Mathematics
Narmada College of Science and
Commerce, Bharuch (Guj.) India

CONTENTS

CHAPTER – 1

INTRODUCTION

INTRODUCTION

In times of national crises, political, economical or cultural, the talents from all walks of life join together to overcome the situation and tide over the crises. These combined efforts always result in new discoveries and techniques. Operations Research is also the outcome of such situations over the last few decades. There were certain scientific methods, considered to be man's greatest assets, which were used only in physical sciences and for technical problems before World War II. The other fields were not considered suitable for the application of these scientific methods as the problems connected with unpredictable human behaviour were not expected to retain the same exactness in precision. But the changes in the size and complexities of human organisations necessiated a more complex and objective study with a scientific basis to solve the problems, e.g. in industrial organisations the division of managerial functions into a set of subfunctions to serve efficiently the interest of whole organisation. The use of scientific methods to study the phenomenon connected with human behaviour has been possible due to the development of nondeterministic techniques like probability and statistics.

The term Operations Research was first coined in the year 1940 by McClosky and Trefthen of U.K. This new science came into existence in military context. During World War II, military management invited a team of scientists from various disciplines to assist in solving strategic and technical problems, i.e., to discuss, evolve and suggest ways and means to improve the execution of various projects in military. By their joint efforts, experiences

and deliberations they suggested certain approaches that showed remarkable progress. This new approach to systematic and scientific study of the operations of the system was called the Operations Research. Following the end of War the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex executive type problems.

During the year 1950, Operations Research achieved recognition as a subject of academic studies in the universities. Since then the subject has been gaining more and more importance for students of Economics, Management, Public Administration, Behavioural Science, Social Work, Mathematics, Commerce and Engineering. In the same year 1950, the Operations Research Society of America was formed in order to increase the impact of O.R. and to establish the rapport between all its students and users. Other countries followed suit and in 1957 the International Federation of O.R. Societies was established. In several countries, International Scientific Journals in Operations Research began to appear in different languages. The primary Journals are Operations Research, Transportation Science, Management Science, Operational Research Quarterly, Journal of Canadian Operational Research Society etc.

In India, Operations Reseach came into existence in 1949 with the opening of an O.R. unit at the Regional Research Laboratory at Hyderabad. At the same time, another group was set up in the Defence Science Laboratory which devoted itself to the problems of stores, purchases and planning. In 1953, an O.R. unit was established in the Indian Statistical Institute, Calcutta for the application of O.R. methods in national planning and survey. O.R. Society of India was formed in 1955, which became a member of International Federation of O.R. Societies in 1959. The first

conference of O.R. Society of India was held in Delhi in 1959. It was felt that the primary necessity of the country is to produce well trained O.R. practitioners who could tackle practical problems. With the time some of the institutions started producing O.R. workers to meet the present and future needs of India. It was also decided to start a Journal of Operations Research, which took practical shape in 1963 in the form of 'Opsearch'. The Indian Institute of Industrial Engineers has also promoted the development of Operations Research, and its journals 'Industrial Engineering' and 'Management' publish several papers related to the area. Other journals which deal with O.R. are : Journal of National Productivity Council, Materials Management, Journal of India and the Defence Science Journal. Towards the applications of O.R. in India, Prof. Mahalonobis made the first important application. He formulated the second five-year plan with the help of O.R. techniques to forecast the trends of demand, availability of resorces and for scheduling the complex schemes necessary for developing our country's economy. It was estimated that India could become self sufficient in food merely by reducing the wastage of food by 15%. O.R. techniques are used to achieve this goal. In the Industrial sector, inspite of the fact that the opportunities of O.R. work at present are very much limited, organised industries in India are gradually becoming conscious of role of Operations Research and good number of them have well trained O.R. teams. With the exception of the government and textile industries, applications of O.R. in other industries have been more or less equally distributed.

The objective of O.R. is to provide a scientific basis to the decision makers for solving the problems involving the interactions of various components of organisation by employing a team of scientists from all disciplines, all working together for finding a solution which is in the best interest of organisation as a whole. The best solution thus obtained is known as

optimal decision.

Operations Research can be considered as a source to other new sciences. Literally, the word 'Operation' may be defined as some action that we apply to some problem or hypothesis and the word 'research' is an organised process of seeking out facts about the same. Though it is difficult to give complete and exhastive definition of O.R., it can be defined as "O.R. is an art of giving bad answers to the problems which otherwise have worse answers". Operations Research has got wide scope. In general whenever there is any problem simple or complicated, the O.R. techniques can be applied to find the best solution. It provides an understanding which gives the expert new insights and capabilities to determine better solutions in his decision making problems with great speed, competence and confidence.

INVENTORY CONTROL

Inventory is defined as any idle resources of an enterprise. It is a physical stock of goods which is kept for the purpose of future affairs at the minimum cost of funds. Inventory generally means raw materials in process, finished products, spares etc. stocked in order to meet an unexpected demand or distribution in future. Though inventory of materials is an idle resources which is not meant for immediate use, it is almost essential to maintain some inventories for the smooth functioning of an organisation or enterprise.

Inventory management of physical goods is an integral part of a system common to all sectors of the economy such as business, agriculture and defence etc. In these systems inventories are maintained to regulate the production process and to protect against the risk of stockouts. An inventory of materials become essential in order to

(i) promote smooth and efficient functioning of an organisation.

(ii) provide an adequate service to the customers.

(iii) take advantage of price discount by bulk purchasing.

(iv) take advantage of batching and longer production runs.

(v) to provide a safeguard for variation in raw material delivery time or lead time to allow flexibility in production scheduling etc.

Though the inventories are essential to provide an alternative to production or purchase in future, they also mean lock up capital of an organisation. Maintenance of inventories also cost money by way of expenses on stores, equipments, insurance etc. Thus excess inventories are not desirable and therefore requires for controlling the inventories in most profitable way. Introduction of a proper inventory control system helps in keeping the investment in the inventories as low as feasible and yet ensures availability of materials by providing adequate protection against uncertaintities of supplies and consumption of materials and allows full advantage of economies of bulk purchase and transportation costs. Such a system in an organisation reduces considerable chances of going out of stock and at the same time leads to the reduction in inventory levels and release of capitals. The later has a direct effect on the profitability of the company's business. Therefore the basic inventory control problem becomes of determining—when should an order for materials be placed and secondly how much should be produced at the beginning of each time interval or what quantity of an item should be ordered each time. Following are the definitions of some costs and terms involved in inventory problems.

Set Up Cost : This is the cost associated with the setting up of machinery before starting the production. Set up cost is generally assumed to be independent of the quantity ordered for or produced.

Ordering Cost : This is the cost associated with ordering of raw materials for production purposes. Advertisements, consumption of stationery

and postage, telephone charges, rent for space used by the purchasing department, travelling, expenditure incurred etc., constitute the ordering cost.

Production Cost : When large production runs are in process, these result in reduction of product cost per unit and also when 'quantity discounts' are allowed for bulk orders for purchase above a certain specified quantity, the order quantity may suitably be adjusted so as to take advantage of these discounts.

Holding Cost : These are the costs associated with the storage of an inventory untill its use or sale. These costs generally include the costs such as rent for space used for storage, interest on many locked-up, insurance of stored equipments, production taxes, depreciation of equipment and furniture etc.

Shortage Cost : These are the costs involved with the inadequacy of the stock of certain commodity to meet the demand for some time. Shortage of stocks may result in the cancellation of orders and heavy losses in sale which in turn may result in loss in good will, profit and even the business itself.

Besides the costs that determine the profitability, other factors which play an important role in the study of inventory problems are—

Demand : Demand is the number of units required per period and may be known either exactly or in term of probability or be completely unknown. Further, if the demand is known, it may be either fixed or variable per unit time. Problems in which demand is known and fixed are called deterministic problems, whereas those problems in which demand is assumed to be random variable are termed as stochastic or probabilistic problems.

Lead time : The time taken between placing an order and its actual arrival in the inventory.

Stock Replenishment : Although an inventory problem may operate with lead time, the actual replacement of stock may occur instantaneously or uniformly. Instantaneous replenishment occurs in case the stock is purchased from outside sources whereas the uniform replenishment may occur when the product is manufactured by the company.

Time Horizon : The time period over which the inventory level will be controlled is called the time horizon. This horizon may be finite or infinite depending upon the nature of demand for the commodity.

Deterioration : It is decay or damage of the item so that it is not meant for its original purpose.

TECHNIQUES

Generally every organisation consumes several items of stores. As all the items are not of equal importance, a high degree of control on inventories of each item is neither practical nor worth while. Therefore, it becomes necessary to classify the items in groups depending upon their importance. Such a classification is known as principle of selective control as applied to inventories. The most common selective techniques are based on ABC (Always better control) analysis, VED (Vital, Essential, Desirable) analysis and FNSD (Fast, Normal, Slow, Dead) classification according to descending order of their consumption value, criticality, inventory value and usage rate respectively. These different selective control techniques are suitable in different situations, which depend upon the nature of the inventories carried out by an organisation to decide which selective control technique must be chosen. Sometimes these techniques are used combindly in order to have better judgement.

CLASSIFICATION OF INVENTORY MODELS

Basically the inventory models are classified in two categories –

(1) Deterministic Inventory Models : In such inventory models,

demand is completely predetermined. These problems are also termed as the economic order quantity problems. The first mathematical deterministic model which we know about is generally referred to as Wilson's formula and was obtained by F.W. Harris in 1915 while he was working on a system of production planning and inventory control on behalf of Washington Co. The problem was to determine the order quantity and the time interval between the successive orders when demand is known, production is instantaneous, shortages are not allowed and lead time is zero. The Wilson's formula is given by

$$Q^{\circ} = \sqrt{\frac{2\,d\,A}{C_1}} \tag{1.1}$$

where d denotes the demand, C_s the set up cost and C_1 the holding cost per unit per unit time. A ?

In the above model discussed, replenishment time is zero, i.e. items ordered are procured in one lot. In practical life this happens rarely particularly in production policies, the items can be produced on a machine at a finite rate per unit of time and hence replenishment time is not zero. Taking all the assumptions same as in above model except that the rate of replenishment is finite, then such model is known as EOQ model with finite rate of replenishment. Inventory models in which demand is satisfied throughout in a system are referred as EOQ model without shortages. The models in which shortages occur for some period in the system are known as EOQ models with shortages. The inventory models in which deterioration (constant or variable) affects the items in inventory of the system are known as EOQ models with deterioration.

(ii) Probabilistic Inventory Models : Inventory models in which demand is not known exactly but the probability distribution of demand is

known are referred as probabilistic or stochastic inventory model. The first probabilistic model, now known as 'news boy problem' was obtained during World War II.

SURVEY OF THE WORK ON INVENTORY CONTROL

Since the development of Wilson's Economic Order Quantity inventory model in 1915, a number of research papers have appeared analysing mathematical models of inventory control under various costs and demand structures and restrictive situations and (or) conditions. A discription and analysis of these models are given by authors of standard text book on inventory control viz, **Arrow, Karlin and Scraf, Naddor** [75], **Tersini** [103] and **Wagner** etc. An up to date account of literature in inventory management is prepared by **Silver** [89], who has also hinted on possible areas for further research.

Traditionally inventory models assume that consumption rate does not depend upon ordered quantity and are uncorrelated with each other. However the above assumption may not be true in many cases. For instance in some particular situation the consumption may go up for higher inventory level and vice-versa.

Gupta and Vrat [50] derived an EOQ model taking dependence of demand rate on ordered quantity for no shortage case when lead time is zero. They established the model for both the cases, i.e. when rate of replenishment is instantaneous and, finite for minimizing the total cost of the system.

In the aboce EOQ model, cost minimization technique is used to take care of stock dependent consumption rate. The expression for variable demand rate is used in the cost equation for constant demand, and total cost for the system has been calculated, which is incorrect except where

the demand rate depends on replenishment size.

Mandal and Phaujdar [72] proposed an EOQ model without shortages where replenishment of stock is instantaneous and demand rate depends upon the current stock status. A linear functional relation $[r(q) = \alpha + \beta q]$ between demand rate r and stock level q is assumed and criteria of maximizing total profit per unit time is applied to remove the drawback left by **Gupta and Vrat** [50]. Condition of optimality for this model is given as

$$A - \left\{C_1 - \beta(p - C)\right\} \frac{(\alpha + \beta S) \log \left(1 + \frac{\beta}{\alpha} S\right) - \beta S}{\beta^2} = 0 \tag{1.2}$$

if the criteria of maximizing the total profit per unit time is utilized, and if criteria of minimization of total average cost per unit time is used then we would obtain optimality condition as

$$A - (C_1 + \beta C) \frac{(\alpha + \beta S) \log \left(1 + \frac{\beta}{\alpha} S\right) - \beta S}{\beta^2} = 0 \tag{1.3}$$

where A is the set up cost, C and p are unit cost and selling price of the item, C_1 is the unit holding cost per unit time. and S is the highest stock level. Moreover, if $\beta \to 0$, both expressions reduce to the classical formula as given by Wilson for uniform demand.

Baker and Urban [13] have analysed a situation assuming the demand rate to be dependent on the on hand inventory according to the relation $r = \alpha q^{\beta}$ where $\alpha > 0$, $0 < \beta < 1$.

Datta and Pal [36] have studied a model where the demand rate declines along with stock level down to a certain level of the inventory, and then the demand rate becomes constant for the rest of the cycle. Such a situation can be seen to occur in cases where the customers arrive

to purchase goods attracted by the huge stock, and after certain level of declining inventory, only a limited number of customers arrive to purchase goods owing to such factors as good will, good quality, genuine price level of the goods, locality of the shop etc.

Donaldson [23] derived an analytical solution to the problem of obtaining the optimal number of replenishments and optimal replenishment time of an EOQ model with a linearly time dependent demand pattern over a finite time horizon. His method needs a lot of computations and is not easy to use. Later many other researchers have proposed various other techniques, which require comparatively less computations to solve the same inventory problem. Among them, **Silver** [90], **Mitra** *et. al.* [68], **Buchanan** [9] and **Ritchie** [85] are worth mentioning. In all these above models, shortages are not taken into account. But when the holding cost is significant as compared to the shortage cost, the total cost for each replenishment cycle may be reduced by allowing shortages for some time.

Zangwill [111] developed a discrete in time dynamic programming algorithm to solve an inventory model by allowing the inventory level to be negative where the demand pattern is time dependent.

Deb and Chaudhuri [30] have attempted to extend the heuristic of **Silver** [90] by allowing shortages which are fully backlogged, but their exposition is incorrect. **Dave** [34] has pointed out the flaws in the article of **Deb and Chaudhuri** [30] and has given the correct analysis. Following the approach of **Donaldson** [23], **Murdeshwar** [71] tried to derive an exact solution for a finite horizon inventory model to obtain at the optimal number of replenishments, optimal replenishment time, assuming the demand rate to be a linear function of time and considering shortages which are completely accumulated. His exact solution turns out to be incorrect as expression for

shortage cost was wrong. **Dave** [35] later, has proposed a method for siolving the same problem reducing the necessary computations. He has also shown that no shortage period for each cycle is a constant fraction of the corresponding cycle length.

Deterioration is one of the important factor to be considered in the study of inventory systems. As defined earlier, deteriorated item does not retain its original value, therefore its effect on inventory system cannot be disregarded.

Ghare and Schrader [41] were the first to introduce the notion of deterioration. They proposed an EOQ model in which items deteriorate at a constant rate. Soon after this, **Emmons** [38] followed a similar exposition for the decay of radioactive nuclear generators. This work was extended by **Covert and Philip** [16] and **Philip** [81] by constructing EOQ models for items with variable rate of deterioration which was further generalised by **Shah** [94] allowing shortages and considering general deteriorating functions.

Shah and Jaiswal [91] developed an order level inventory model for constantly deteriorating items where a constant fraction of on hand inventory deteriorate each unit of time. The model is established for both the cases, the deterministic as well as probabilistic. But they committed a mistake while calculating average inventory holding cost for the system. Authors have actually considered the inventory level of the system exactly linear in the prescribed interval which is not correct. A similar type of error has also been made in the development of the probabilistic model. The total cost of the system as given by **Shah and Jaiswal** [91] is

$$K = \frac{C}{T}(S - rt_1) + \frac{C_1 St_1}{2T} + \frac{C_2 r}{2T}(T - t_1)^2 \tag{1.4}$$

where the symbols have their usual meanings.

S.P. Aggarwal [1] developed an order level inventory model for a system with constant rate of deterioration by rectifying the error in Shah and Jaiswal's analysis in deetermining the average inventory holding cost in the model.

The term $\frac{C_1 S t_1}{2T}$ in the right hand side of equation (1.4) has been taken as the total average inventory holding cost per unit time. It is clear that they have assumed the inventory level of the system to be linear in $(0, t_1)$. The corrected average inventory holding cost per unit time should be

$$\frac{C_1}{T} \int_0^{t_1} \left[S - \frac{r}{\theta} \left\{ \exp(\theta t) - 1 \right\} \exp(-\theta t) \right] dt \qquad (1.5)$$

and the corrected total cost equation of the system then becomes

$$K = \frac{c\theta S^2}{2rT} + \frac{C_1 S^2}{2rT} \left[1 - \frac{2\theta S}{3r} \right] + \frac{C_2 r}{2T} \left[T - \frac{S}{r} \right]^2 + \frac{C_2 \theta S^2}{2rT} \left[T - \frac{S}{r} \right] \qquad (1.6)$$

where symbols have their usual meanings.

Observing both the cost equations (1.4) and (1.6), it is interesting to note that assuming linearity the multiplier 2/3 in the second term is not present which means that assuming linearity under-estimates the inventory holding cost.

Dave and Patel [26] have derived a (T, S_i) policy inventory model for time proportional demand. In this model demand rate is linearly changing with time and deterioration is taken to be constant, the planning horizon is known and finite and replenishment periods are assumed to be constant. They have found the optimal number of replenishments which are instantaneous.

In all these models, discussed so far, time is treated as a continuous

variable. **Dave** [24, 25] has presented two models for deteriorating items, where time is a discrete variable.

Dave [27] proposed an order level inventory model for continuously deteriorating items. The model is continuous in units but allows discrete opportunities for replenishments. The demand is assumed to vary from time unit to time unit and occurs instantaneously at the beginning of each time unit. The deterioration is a constant fraction of on hand inventory at the beginning of a time unit, which also varies from time unit to time unit. Lead time is zero and shortage are allowed which are fully backlogged.

Sachan, R.S. [98] extended the model developed by **Dave and Patel** [24] for deteriorating items with time proportional demand. In this inventory model shortages were not allowed. **Sachan** [98] has derived a (T, S_i) policy inventory model by assuming demand as time dependent and permitting shortages which are completely accumulated. He has also corrected the numerical results and explained the potential cost savings in comparision with the model given by **Dave and Patel** [24].

Datta and Pal [32] have presented an EOQ model considering the existence of suitable power demand pattern and special form of Weibull two parameters density function. Such a special form of deterioration is chosen in order to make the problem more tractable. The model allows shortages for a prescribed period of time. Both the cases, deterministic as well as probabilistic have been studied in this model.

Mandal and Phaujdar [72] has utilized cost minimization criteria to put the total average cost of the system to a minimum when the items deteriorate at a variable rate, production rate is uniform, shortages occur, set up cost is considered and demand rate is dependent on instantaneous inventory level.

Goswami and Chaudhuri [53] formulated an inventory replenishment policy over a fixed planning period for a deteriorating item having a deterministic demand pattern with a linear trend and allowing shortrarrges. The number of re-orders, the interval between two successive reorders and the shortage intervals over a unit time horizon are all determined in an optimal manner so as to minimize the total average system cost. In the last, effects of variation in the deterioration on the optimal policy are also explained with a numerical example.

It is usually assumed in inventory models that the ware house owned by the management is adequate to stock the inventory procured. However, it can be attracted to the management to purchase or produce more goods that can be stored in the owned ware house (OW), excess quantities are stored in rented ware house (RW) which is located above from OW. The management resorts to this practice either when the acquisition costs are higher than the holding costs in RW or when it gets an attractive price discount for bulk purchase. The consumers are served only from OW as inventory holding charges in RW are higher than those in OW, and to reduce the holding cost, stocks of RW are cleared first. Inventory models dealing with such situations are termed as 'Inventory models with two levels of storage'.

A model of above type was first developed by **Hertely** [58]. He derived a computational procedure for determining the optimal order quantity under the assumption that transporting a unit from RW to DW is not significantly high.

Sharma [99] presented a solution procedure for the deterministic inventory model with two levels of storage taking transportation cost of a unit from RW to OW into account.

Goswami and Chaudhuri [54] derived a deterministic inventory model in which demand varies over time with a positive linear trend. Here the transportation cost depends on the quantity to be transported unlike **Sharma** [99] in which this cost was a fixed constant independent of the quantity being transported.

Whenever a supplier gives an advance notice to the buyer of an increase in the unit purchase price of a product, the buyer can very often take advantage of the last few opportunities of buying the product at a lower unit price.

The decision problem facing the buyer is to determine the economic ordering policy during time period the product can be purchased at a lower unit price. The problem of determining economic ordering policy under the conditions of announced price increase has attracted the attention of many researches.

Let t_a, t_b and t_c be the timings of the price increase, at which stock held at time t_a reaches to zero. and at which the price increase is to be effective.

Naddor [75] has considered the case when $t_a = t_b$, and determined the economic order quantity assuming that buyer has only one opportunity to place a special order at the end of the EOQ cycle.

Taylor and Bradley [104], **Tersini** [103]. **Lev and Soyster** [63] have considered the situation when $t_c < t_b$, when no replenishment order is scheduled before t_c.

In case $t_c > t_b$, at least one replenishment is scheduled during the time interval (t_c, t_b). The buyer has time to adjust order quantities during this interval. **Taylor and Bradley** [104], **Goyal** [45], and **Lev and Soyaster**

[63] first determined an integer number of purchase orders of equal size during the interval (t_c, t_b) and then placed a special order just at t_c. Taylor and Bradley defined such a policy as the Modified Lot Size Strategy (MLS Strategy).

Goyal and Bhatt [51] proposed a generalized MLS policy which renders the MLS policies of **Taylor and Bradley** [104] as a particular cases. They assumed n purchase orders of equal sizes placed prior to the (n + 1) the special order which is placed before or at t_c and after the special pruchase order, the usual EOQ according to the new unit purchase price is implemented.

Goyal, S.K. [52] also investigated a simple procedure for determining the optimal ordering policy for an item for which an increase in the price has been announced. With this technique, need for evaluation and comparision of the total cost of alternate ordering policies for determining optimal ordering policy is not required.

Krishnamoorthy and Manoharan [61] have considered an inventory system with unit demand, varying ordering levels and random lead times. Ordering level is determined by the number of demands during last lead time when backlog is not permitted. The time dependent probability distribution of the inventory level is obtained and correlation between the number of demands during a lead time and the length of next inventory dry period is established.

Yanasse [110] studied an EOQ system in which an anticipated price may increase and determination of exceptional lot is required. He used the criteria of minimizing the maximum errors in terms of cumulative costs to determine this ordered quantity, which avoids the pitfalls raised on certain

works that compare out of phase inventory policies.

Zodeh and Lee [74] considered an inventory system where orders may arrive in two shipments, that is, in the first shipments order may contain only part of itmes ordered, while the rest of items arrive in a second shipment. However, the amount of first shipment is random. Such situation arise when the supplier partially fills the order if it does not have significant stock to satisfy the amount demanded in an order, or when an arriving order contains some deffective items that can be returned to and replaced by the supplier at some later date. He has determined the approximate cost function, ordered quantity and reorders points that minimize it.

Altiok [4] considered an (R, r) continuous inventory policy in a production or inventory environment consisting of a production facility and a finished product warehouse for one type of product. Policy indicates that production starts when on hand stock reaches to r and continues untill the stock level becomes R. Here demand process is assumed to be compound poisson and processing times are phase type. The problem of finding cost minimizing values of r and R is studied through the analysis of inventory process for both back ordering and lost sales cases.

SUMMARY OF THE THESIS

In this thesis an endeavour is made to establish some more realistic deterministic and probabilistic inventory models under different conditions.

Chapter first, is of introductory nature, this provides history, evolution and development of Operations Research. Some methods and teachniques involved in the study of inventory control have been out lined. Various costs and terms related to the inventories are explained and different types of

models are discussed depending upon the conditions and assumptions prescribed. A general literature survey on inventory control is also given. Thus, this chapter motivates for the work, accomplished in this thesis.

In **second chapter**, and EOQ model is derived, taking the dependence of stock quantity on the demand rate. The model is first established for instataneous case of replenishment and then finite replenishment case has been considered. Different forms of the functional relationships existing between the demand rate and stock quantity have been taken to derive the EOQ model, using cost minimization criteria.

In above EOQ model, while evaluating total cost per unit time of the system, expression for variable demand rate has been substituted in the cost equation obtained for constant demand rate, which is not justified as this would not consider the stock dependent consumption rate except where the demand depends upon the replenishment size. In **chapter third**, keeping this in view, an EOQ model is formulated and profit maximization technique is employed to obtain the optimal solution. The expressions for EOQ model are derived when rate of replenishment is instantaneous and shortages are not allowed.

Chapter fourth is devoted to the study of EOQ models where items in inventory deteriorate at some constant rate. The EOQ model is developed assuming different functional relationships between the demand rate and current stock level and replenishment of the stock is instantaneous.

The optimum cycle length and total maximum profit per unit time can be obtained, if the explicit form of demand rate is known. It is also deduced that if there be no deterioration, the result derived in first case reduce to those given by **Mandal and Phaujdar** [72].

In **fifth chapter**, an order level inventory model for deteriorating items is produced. There were some remarkable errors in the order level inventory model given by **Mitali Roy Choudhary and K.S. Chaudhary** [19]. These errors are modified here and corrected results are obtained. In this model units in inventory procure at a finite rate r, items consume at a rate r (k r), shortages are allowed and completely backlogged, lead time is zero and deterioration is a constant fraction of the on hand inventory. The model is discussed for both deterministic and probabilistic demands. A numerical problem is also worked out to determine optimal value of stock quantity, timings of sub- intervals and total minimum cost for the system.

In inventory models, developed by various anthors, demand rate either depends on stock quantity or time. **Chapter six** deals with the inventory models considering the dependence of demand rate on stock quantity and time both. The model is derived for instantaneous replenishment case not allowing shortages.

In **chapter seven**, an inventory model is developed where demand varies exponentially with time. It is assumed that rate of deterioration is constant and production occurs as soon as the stock level reaches to zero, lead time is zero and shortages are permitted for a fixed time interval. Later EOQ model is discussed for non-deteriotation items and it is shown that when no deterioration takes place. former EOQ model reduce to the later. Probabilistic cases of above models are also studied in this chapter.

Chapter eight is the generalization of the EOQ model developed in previous chapter. In the inventory model developed here, demand rate is exponential function of time and rate of deterioration is variable i.e. $\theta = \theta_0 t$ where θ_0 is a constant and $0 < \theta < 1$, $t > 0$. Deterministic as well

as probabilistic models are considered with backlogging option which are fully accumulated. It is deduced that when there be no deteriration the EOQ model reduces to the model for non-deteriorating items presented in seventh chapter.

In the **last chapter** a (T, S_i) policy inventory model is determined with linear trend in demand rate. The model is formulated for variable rate of deterioration of items in the system. Here the number of replenishments periods, scheduling time, length of the cycle and total cost are, all determinied in an optimal manner. In the second part of the chapter this model also incarporates shortages. Finally, it is shown that if deterioration does not occur, both of these models reduce to the corresponding models for non-deteriorating items as obtained by **Naddor** [75].

CHAPTER – 2

AN EOQ MODEL WITH STOCK DEPENDENT DEMAND RATE

INTRODUCTION

In conventional inventory models it is assumed that demand (consumption) rate does not depend upon inventory level. But this assumption is unlikely to be true in the cases where the consumption pattern is likely to be influenced by the stock maintained. For example in some particular situation the consumption may increase if the level of inventory is high and vice-versa, because the customers may be motivated with the ease of availability or abundance of supply of the items. This behaviour of the customer results the case of stock dependent demand pattern. In such cases the total system cost should also include the direct purchase cost of the items as the inventory prices do affect the direct purchase cost of the product.

Gupta and Vrat [50] presented an EOQ model where stock level affects the consumption rate. They established expressions for EOQ models for different functional relationship with instantaneous rate of replenishment and with no shortages. They also obtained expressions for finite rate of replenishment with no shortages and linear consumption rate.

In this chapter an EOQ model is derived with stock status dependent consumption rate. Expressions for EOQ have been established with instantaneous rate of replenishment and finite rate of replenishment respectively in no shortages case for different functional relationships.

ANALYSIS

INSTATANEOUS REPLENISHMENT CASE

In classical EOQ model for constant demand rate r, the total cost of the system per unit time is given by

$$K(q) = rC + \frac{Ar}{q} + \frac{CC_1q}{2} \tag{2.1}$$

Where A denotes set up cost per order, C the unit purchase cost, C_1 unit carrying charges per unit per unit time, q the order quantity and T the planning period. T is not r ?

EOQ model has been considered for the following functional relations–

(i) $r(q) = \alpha + \beta \log q$

(ii) $r(q) = \alpha + \beta a^{q}$

(iii) $r(q) = \alpha + \beta a^{-q}$?

(iv) $r(q) = \alpha + \beta q + \gamma q^2$

where α, β, γ and a are positive constants.

Case (i) : $r = \alpha + \beta \log q$

Substituting this value of demand rate r in equation (2.1), we get

$$K(q) = \alpha C + \beta C \log q + \frac{\beta A \log q}{q} + \frac{CC_1q}{2} + \frac{\alpha A}{q} \tag{2.2}$$

Now, for cost K to be minimum, condition is $\frac{dK(q)}{dq} = 0$, which on using equation (2.2) gives,

$$\frac{CC_1}{2} + \frac{\beta C}{q} + \frac{BA}{q^2}(1 - \log q) - \frac{\alpha A}{q^2} = 0 \tag{2.3}$$

On simplifying this gives,

$$CC_1q^2 + 2\beta Cq + 2\beta A(1 - \log q) = 2\alpha A$$

or $$CC_1q^2 + 2\beta Cq - 2\beta A\log q + 2A(\alpha + \beta) = 0 \tag{2.4}$$

The above equation can be solved by Newton-Raphson method for obtaining the value of q.

Also $$\frac{d^2K}{dq^2} = \frac{A\left[\beta(\log q^2 - 3) + 2\alpha\right] - \beta Cq}{q^3} \tag{2.5}$$

which is positive for all values of q given by

$$\frac{\beta\,(\log q^2 - 3) + 2\alpha}{q} > \frac{C\beta}{A} \tag{2.6}$$

Thus equation (2.4) gives optimal value of q under condition (2.6).

Case (ii) : $r = \alpha + \beta a^q$

Substituting this value of r in equation (2.1), we get

$$K(q) = \alpha C + \frac{CC_1q}{2} + \frac{\alpha A}{q} + \beta Ca^q + \frac{A\beta a^q}{q} \tag{2.7}$$

For optimal cost we have

$\frac{dK}{dq} = 0$, which gives,

$$\frac{CC_1}{2} - \frac{\alpha A}{q^2} + \beta Ca^q \log a + \frac{\beta Aa^q \log a}{q} - \frac{\beta Aa^q}{q^2} = 0 \tag{2.8}$$

or $$CC_1q^2 - 2\alpha A + 2\beta C \log a\, q^2a^q + 2\beta A\,(\log a)\, a^q - 2\beta Aa^q = 0 \tag{2.9}$$

Equation (2.9) can be solved by Newton-Raphson method for q.

Also $$\frac{d^2K}{dq^2} = \frac{2\alpha A + \beta C\,(\log a)^2q^3a^q + A\beta a^q\left[1 + \left\{(\log a)\,q - 1\right\}^2\right]}{q^3} \tag{2.10}$$

This shows that $\frac{d^2K}{dq^2} > 0$ for all values of q. Thus equation (2.9) gives the optimal value of q.

If we take $a = e$ then the result reduces to that of **Gupta and Vrat** [50].

Case III : $r = \alpha + \beta a^{-q}$

Putting this value of r in equation (2.1), we get

$$K(q) = \alpha C + \frac{CC_1 q}{2} + \frac{\alpha A}{2} + \beta C a^{-q} + \frac{A\beta a^{-q}}{q} \tag{2.11}$$

For optimality of q, we have

$$\frac{dK}{dq} = \frac{CC_1}{2} - \frac{\alpha A}{q^2} - \beta C a^{-q}(\log a) - \frac{\beta A a^{-q}(\log a)}{q} - \frac{\beta A a^{-q}}{q^2} = 0 \tag{2.12}$$

which simplifies to

$$CC_1 q^2 - 2\alpha A - 2\beta C(\log a)q^2 a^{-q} - 2\beta A(\log a)q a^{-q} - 2\beta A a^{-q} = 0 \tag{2.13}$$

Equation (2.13) can be solved by Newton-Raphson method for q.

Also $\dfrac{d^2K}{dq^2} = \dfrac{1}{q^3}\left[2\alpha A + \beta C(\log a)^2 q^3 a^{-q} + A\beta a^{-q}\left\{1 + (q\log a + 1)^2\right\}\right] > 0$

(2.14)

for all values of q.

Therefore equation (2.13) gives the optimal value of q.

Case (iv) : $r = \alpha + \beta q + \gamma q^2$

Using this value of r in equation (2.1), we get

$$K(q) = \alpha C + A\beta + \left(\beta C + A\gamma + \frac{CC_1}{2}\right) q + \gamma C q^2 + \frac{A\alpha}{q} \tag{2.15}$$

For total cost to be minimum

$\dfrac{dK}{dq} = 0$, which implies,

$$\beta C + A\gamma + \frac{CC_1}{2} + 2\gamma C q - \frac{A\alpha}{q^2} = 0 \tag{2.16}$$

On simplifying, above expression reduces to

$$2\gamma Cq^3 + \left(\beta C + A\gamma + \frac{CC_1}{2}\right)q^2 - A\alpha = 0 \tag{2.17}$$

Also $$\frac{d^2K}{dq^2} = 2\gamma C + \frac{2A\alpha}{q^3} \tag{2.18}$$

It is clear that $\frac{d^2K}{dq^2} > 0$ for all values of q. Therefore solution of equation (2.16) is the optimal value of q.

FINITE REPLENISHMENT CASE

In this section EOQ has been established under finite rate of replenishment for the following functional relations–

(i) $r' = r + \frac{\beta}{q}$

(ii) $r' = r + \beta q^{\gamma}$

(iii) $r' = r + \beta e^{q}$

where β and γ are positive constants and λ' is the consumption rate during depletion time. The total cycle time T consists of two parts; T_{rp} the time during which replenishments come and T_d the time during which stock depletion takes place. Thus $T = T_{rp} + T_d$. The finite replenishment rate and the consumption rate are taken k and r respectively such that $k > r$. Obviously net rate of procurring of items in inventory is $k - r$.

Case (i) : Following **Gupta and Vrat** [50] the inventory carrying cost per cycle is given by

$$\frac{CC_1q^2}{2r^*}\left(1 - \frac{r}{k}\right) \text{ where}$$

$$\frac{1}{r^*} = \frac{1}{k} + \frac{1}{r}\left(1 - \frac{r}{k}\right)$$

The total cost per unit time of the system is

$$K(q) = \frac{1}{T}\left[A + \frac{CC_1 q^2}{2r^*}\left(1 - \frac{r}{k}\right)\right]$$

$$= \frac{Ak\beta}{q(\beta + kq)} + \frac{Akr}{(\beta + kq)} + \frac{CC_1 q}{2}\left(1 - \frac{r}{k}\right) \tag{2.19}$$

For which optimal condition is

$$\frac{dK}{dq} = \frac{-Ak\beta}{q^2(\beta + kq)} - \frac{Ak^2\beta}{q(\beta + kq)^2} - \frac{Ak^2\lambda}{(\beta + kq)^2} + \frac{CC_1}{2}\left(1 - \frac{r}{k}\right) = 0 \tag{2.20}$$

This equation can be solved for q by Newton-Raphson method.

Also $$\frac{d^2K}{dq^2} = \frac{2Ak\beta}{q^3(\beta + kq)} + \frac{2Ak^2\beta}{q^2(\beta + kq)^2} + \frac{2Ak^3\beta}{q(\beta + kq)^3} + \frac{2Ak^3 r}{(\beta + kq)^3} \tag{2.21}$$

Clearly $\frac{d^2K}{dq^2} > 0$ for all values of q.

This solution of (2.20) gives the optimal value of q.

Case (ii) : $r' = r + \beta q^\gamma$

Here the total cost per unit time is given by

$$K(q) = \frac{Akr}{q(k + \beta q^\gamma)} + \frac{Ak\beta q^{\gamma - 1}}{(k + \beta q^\gamma)} + \frac{CC_1 q}{2}\left(1 - \frac{r}{k}\right) \tag{2.22}$$

Condition for which to be optimal is

$$\frac{dK}{dq} = -\frac{Akr}{q^2(k + \beta q^\gamma)} - \frac{Akr\beta\gamma q^{\gamma - 1}}{q(k + \beta q^\gamma)^2} + \frac{Ak\beta(\gamma - 1)q^{\gamma - 2)}}{(k + \beta q^\gamma)} - \frac{Ak\beta^2\gamma q^{2\gamma - 2}}{(k + \beta q^\gamma)^2}$$

$$+ \frac{CC_1}{2}\left(1 - \frac{r}{k}\right) = 0 \tag{2.23}$$

Equation (2.23) can be solved by Newton Raphson method for q.

Also $$\frac{d^2K}{dq^2} = \frac{2Akr}{q^3(k+\beta q^\gamma)} + \frac{Ak\beta(\gamma-1)(\gamma-2)q^{\gamma-3}}{(k+\beta q^\gamma)}$$

$$+ \frac{Ak\beta\gamma q^{\gamma-3}\left[r(3-\gamma) - 3\beta(\gamma-1)q^\gamma\right]}{(k+\beta q^\gamma)^2} + \frac{2Ak(\beta\gamma)^2 q^{2\gamma-3}(r+\beta q^\gamma)}{(k+\beta q^\gamma)^3} \tag{2.24}$$

Now $\frac{d^2K}{dq^2} > 0$ if

$$r(3-\gamma) - 3\beta(\gamma-1)q^\gamma > 0$$

or $$q < \left[\frac{r(3-r)}{3\beta(\gamma-1)}\right]^{1/\gamma} \quad \text{and} \quad 2 \le \gamma \le 3 \tag{2.25}$$

Hence solution of equation (2.23) under condition (2.25) gives the optimal value of q.

Caes (iii) : $r' = r + \beta e^q$

In this case the total cost per unit time of the system is

$$K(q) = \frac{Ak(r+\beta e^q)}{q(k+\beta e^q)} + \frac{CC_1 q}{2}\left(1 - \frac{r}{k}\right) \tag{2.26}$$

The optimality condition for q, implies

$$\frac{dK}{dq} = \frac{Ak\beta e^q}{q(k+\beta e^q)} - \frac{Ak(r+\beta e^q)}{q^2(k+\beta e^q)} - \frac{Ak\beta(re^q+\beta e^{2q})}{q(k+\beta e^q)^2} + \frac{CC_1}{2}\left(1-\frac{r}{k}\right) = 0 \tag{2.27}$$

Solution to which can be obtained by Newton-Raphson method.

Also $$\frac{d^2K}{dq^2} = \frac{Ak\beta e^q\left[(q-1)^2+1\right]}{q^3(k+\beta e^q)} + \frac{2Akr}{q^3(k+\beta e^q)} + \frac{2Ak\beta(re^q+\beta e^{2q})}{q^2(k+\beta e^q)^2}$$

$$+ \frac{Ak\beta e^q(\Phi - r)(k-\beta e^q)}{q(k+\beta ew^q)^3} \tag{2.28}$$

It is clear that $\frac{d^2K}{dq^2} > 0$, if $(k - \beta e^q) > 0$ or $q < \log\frac{k}{\beta}$ (2.29)

Solution of equation (2.27) under condition (2.29) gives the optimum value of order quantity q.

DISCUSSION

In EOQ model formulated here, demand rate depends upon the stocks and shortages are not allowed. First model is discussed for instantaneous case of replenishment and later for finite replenishment case for assumed demand rates. A solution procedure is explained to get optimal value of stock quantity and substituting this in cost equation, total cost of the system per unit time can be estimated.

Model derived in second case reduces to the corresponding model given by Gupta and Vrat [50] under specific conditions. Some more functional relations between the demand rate and stock quantity can be constructed to obtain the new results.

CHAPTER – 3

INVENTORY MODELS FOR STOCK DEPENDENT DEMAND RATE

(Different Approach)

INTRODUCTION

EOQ models derived in the previous chapter use cost minimization criteria considering the stock dependent consumption rate for different functional relationships. The expression for assumed variable demand rate has been substituted in the total cost per unit time derived under the assumption of constant demand. Naturally this could not take care of stock-dependent demand rate except where the demand rate is depending on replenishment size.

Mandal and Phaujdar [72] suggested an EOQ model through profit maximization criteria considering the consumption rate depending upon the current stock level to yield the optimal solution, which is more realistic. They assumed a linear dependence of the demand rate on the current stock level.

In this chapter EOQ models are derived with some more plausible functional relationships existing between the demand rate and current stock level.

THE MODEL

Here the EOQ model is established in the case where the replenishment of the stock is instantaneous, shortages are not allowed and the demand rate depends upon the current stock level. The total profit per unit time during time T, obtained by **Mandal and Phaujdar** [72] is

$$Z(S) = \frac{pS - \left[A + CS + C_1 G(S)\right]}{F(S)} \tag{3.1}$$

where p denotes unit selling price of the item, A the set up cost, C unit cost price of the item, C_1 unit holding cost per unit of the item and S is the highest stock level.

Also $$G(S) = \int_0^S \frac{q\,dq}{r(q)} \tag{3.2}$$

and $$F(S) = \int_0^S \frac{dq}{r(q)} = T \tag{3.3}$$

The optimum value of S for total maximum profit per unit time is a solution of $Z'(S) = 0$, provided $z''(S) < 0$ for that value of S. Thus for optimal value S, expression (3.1) implies,

$$F(S)\left[(p - C) - C_1 G'(S)\right] = F'(S)\left[(p - C)S - A - C_1 G(S)\right] \tag{3.4}$$

where prime denotes derivative with respect to S. Equation (3.4) is in general a non-linear equation which can be solved numerically by Newton-Raphson method, if the explicit form of r(q) is known. The optimal cycle length is given by $F(S^*)$ where S^* is the optimal value of S.

ANALYSIS

EOQ model has been established for the following functional relations–

(i) $r(q) = \alpha + \beta q + \gamma q^2$

(ii) $r(q) = \alpha + \dfrac{\beta}{q}$

(iii) $r(q) = \alpha e^{\beta q}$

(iv) $r(q) = \alpha e^{-\beta q}$

(v) $r(q) = \alpha q^{-\beta}$

where α, β and γ are non-negative constants.

Case (i) : $r(q) = \alpha + \beta q + \gamma q^2$

Putting the value of r in equation (3.3), we get

$$F(S) = \int_0^S \frac{dq}{\alpha + \beta q + \gamma q^2}$$

$$= \frac{1}{N} \log \left(\frac{2S\gamma + \beta - N}{2S\gamma + \beta + N}\right) \left(\frac{\beta + N}{\beta - N}\right) \tag{3.5}$$

where $N^2 = \beta^2 - 4\alpha\gamma > 0$

Therefore

$$F'(S) = \frac{4\gamma}{(2S\gamma + \beta)^2 - N^2} \tag{3.6}$$

Also putting the value of r in equation (3.2), we get

$$G(S) = \int_0^S \frac{q\,dq}{\alpha + \beta q + \gamma q^2}$$

$$= \frac{1}{2\gamma} \log \left(\frac{\alpha + \beta S + \gamma S^2}{\alpha}\right) - \frac{\beta}{2\gamma N} \log \left(\frac{2S\gamma + \beta - N}{2S\gamma + \beta + N}\right) \left(\frac{\beta + N}{\beta - N}\right) \tag{3.7}$$

So, $$G'(S) = \frac{\beta + 2\gamma S}{2\gamma\,(\alpha + \beta S + \gamma S^2)} - \frac{2\beta}{(2\gamma S + \beta)^2 - N^2} \tag{3.8}$$

Now substituting the values of F(S), F′(S), G(S) and G′(S) in equation (3.4) and simplifying the resulting expression, we get

$$A + \frac{C_1}{2\gamma} \log \left(\frac{\alpha + \beta S + \gamma S}{\alpha}\right)^2 + \frac{1}{4\gamma N} \log \frac{\left(1 + \frac{2S\gamma}{\beta - N}\right)}{\left(1 + \frac{2S\gamma}{\beta + N}\right)} \times$$

$$\left[(p - C)\left\{(\beta + 2\gamma S)^2 - N^2\right\} - \frac{C_1(\beta + 2\gamma S)\left\{(\beta + 2\gamma S)^2 - N^2\right\}}{2\gamma(\alpha + \beta S + \gamma S^2)} + 2C_1\beta\right] = 0$$

(3.9)

It can be shown that if $\gamma = 0$, the above expression (3.9) reduces to that obtained by **Mandal and Phaujdar** [72].

Equation (3.9) is a transcedental equation which can be solved by Newton Raphson method. The solution of equation (3.9) which satisfies $Z''(S) < 0$ gives the optimal value of S.

Case (ii) : $r(q) = \alpha + \frac{\beta}{q}$

Substituting this value of r in equation (3.3), we get

$$F(S) = \int_0^S \frac{q\,dq}{\alpha q + \beta}$$

$$= \frac{1}{\alpha}\left[S - \frac{\beta}{\alpha}\log\left(1 + \frac{\alpha}{\beta}S\right)\right] \qquad (3.10)$$

Therefore, $$F'(S) = \frac{S}{(\alpha S + \beta)} \qquad (3.11)$$

Also from equation (3.2),

$$G(S) = \int_0^S \frac{q^2}{\alpha q + \beta}\,dq$$

$$= \frac{S^2}{2\alpha} - \frac{\beta}{\alpha^2}S + \frac{\beta^2}{\alpha^3}\log\left(1 + \frac{\alpha}{\beta}S\right) \qquad (3.12)$$

So, $$G'(S) = \frac{S^2}{(\alpha S + \beta)} \qquad (3.13)$$

Substituting values of F(S), F′(S), G(S) and G′(S) from equations (3.10), (3.11), (3.12) and (3.13) respectively in equation (3.4) and simplifying this gives,

$$A + \frac{\beta}{\alpha}\left[\frac{C_1}{\alpha} - \frac{(p - C)}{S}\right]\left[\frac{1}{\alpha}(\alpha S + \beta)\log\left(1 + \frac{\alpha}{\beta}S\right) - S\right] - \frac{C_1 S^2}{2\alpha} = 0 \quad (3.14)$$

If we take $\beta = 0$, then equation (3.14) reduces to

$$A - \frac{C_1 S^2}{2\alpha} = 0 \tag{3.15}$$

which gives $$S^* = \left(\frac{2A\alpha}{C_1}\right)^{1/2} \tag{3.16}$$

This is the classical EOQ formula with uniform demand rate.

Equation (3.14) is a transcedental equation which can be solved by Newton-Raphson method. The solution of equation (3.14) which satisfies $Z''(S) < 0$ gives the optimum value of S.

Case (iii) : $r(q) = \alpha e^{\beta q}$

For this value of r, using equations (3.2) and (3.3), we obtain

$$F(S) = \frac{1}{\alpha\beta}\left(1 - e^{-\beta S}\right) \quad \text{and} \tag{3.16}$$

$$G(S) = -\frac{1}{\alpha\beta} S\, e^{-\beta S} - \frac{1}{\alpha\beta^2} e^{-\beta S} + \frac{1}{\alpha\beta^2} \tag{3.17}$$

Differentiating equations (3.16) and (3.17) with respect to S, we get

$$F'(S) = \frac{1}{\alpha} e^{-\beta S} \tag{3.18}$$

and $$G'(S) = \frac{S}{\alpha} e^{-\beta S} \tag{3.19}$$

Putting these values of F(S), G(S), F′(S) and G′(S) from equations (3.16), (3.17), (3.18) and (3.19) respectively in condition (3.4) for optimality, this gives,

$$A + \frac{C_1}{\alpha\beta^2}\left(1 - \beta S - e^{-\beta S}\right) - (p - C)\left[S + \frac{\left(1 - e^{\beta S}\right)}{\beta}\right] = 0 \tag{3.20}$$

which is a transcedental equation and can be solved using Newton-Raphson method. This solution of equation (3.20) satisfying $Z'' < 0$ gives the optimal value of S.

Case (iv) : $r(q) = \alpha e^{-\beta q}$

With this value of r, expressions for F(S), F′(S), G(S) and G′(S) are as follows:

$$F(S) = \frac{1}{\alpha\beta}\left(e^{\beta S} - 1\right) \tag{3.21}$$

$$F'(S) = \frac{1}{\alpha} e^{\beta S} \tag{3.22}$$

$$G(S) = \frac{1}{\alpha\beta} S\, e^{\beta S} - \frac{1}{\alpha\beta^2} e^{\beta S} + \frac{1}{\alpha\beta^2} \tag{3.23}$$

and $$G'(S) = \frac{S}{\alpha} e^{\beta S} \tag{3.24}$$

Substituting these values of F(S), F′(S), G(S) and G′(S) in equation (3.4) for optimality of S, we obtain,

$$A + \frac{C_1}{\alpha\beta^2}\left(1 + \beta S - e^{\beta S}\right) - (p - C)\left[S - \frac{\left(1 - e^{-\beta S}\right)}{\beta}\right] = 0 \tag{3.25}$$

The solution S of transcedental equation (3.25), which satifies $Z''(S) < 0$ gives the optimum value.

Case (v) : $r(q) = \alpha q^{-\beta}$

Using this value equations (3.2) and (3.3) imply,

$$F(S) = \frac{1}{\alpha(1 + \beta)} S^{1+\beta} \tag{3.26}$$

and $$G(S) = \frac{1}{\alpha(2 + \beta)} S^{2+\beta} \tag{3.27}$$

Differentiating above equations (3.26) and (3.27), we have,

$$F'(S) = \frac{S}{\alpha} \tag{3.28}$$

and $$G'(S) = \frac{S^{1+\beta}}{\alpha} \tag{3.29}$$

Optimal condition (3.4) using equations (3.26), (3.27), (3.28) and (3.29)

reduces to

$$A - \frac{S}{(1+\beta)}\left[(p - C)\beta + \frac{C_1 S^{1+\beta}}{\alpha(2+\beta)}\right] = 0 \tag{3.30}$$

Which is a non-linear equation and can be solved for S by Newton-Raphson method. It can be shown that $Z'' < 0$ for this value of S. Thus, solution of equation (3.30) gives the optimum value of S.

DISCUSSION

Here EOQ models are derived in which production of items in the system is instantaneous and shortages do not occur. Profit maximization criteria is employed to yield the maximum profit per unit time of the system. Some functional forms of demand rate are taken in order to formulate the model.

In some cases, when conditions on demand rate are used, the model reduces to the corresponding already established inventory model for that demand rate. The model presented here can be further extended for finite rate of replenishment and (or) allowing shortages.

CHAPTER – 4

INVENTORY MODEL WITH STOCK DEPENDENT DEMAND RATE AND CONSTANT RATE OF DETERIORATION

INTRODUCTION

The effect of deterioration is very important in many inventory system. Deterioration is defined as decay, damage or spoilage such that the item cannot be used for its original purpose. Food items, drugs, photographic films, pharmaceuticals and radioactive substances are examples of items in which sufficient deterioration can take place during the normal storage period of units and consequently this loss must be taken into account while analysing the system.

Inventory models for perishable or deteriorating items subject to exogeneous demands have been studied by various authors. Efforts in analysing mathematical models of inventory in which a constant or variable proportion of the on hand inventory deteriorates with time have been undertaken by **Ghare and Schrader** [41], **Goyal and Aggarwal** [47], **Covert and Philip** [16], **Shah** [94], **Misra** [67] to name a few.

This chapter deals with inventory model with instantaneous stock replenishment, demand rate depending upon current stock-level and, a constant fraction θ of on hand inventory also deteriorates with time. Profit maximization technique is employed to yield the optimal solution. The functional relationships of demand rate are assumed to discuss the EOQ model.

THE EOQ MODEL

If q is the inventory level at time t, θ is the rate of deterioration and r(q) is the stock dependent demand rate then differential equation

---

The matter of this chapter has been published in Acta Ciencia Indica, Vol. No. 4M, 420 (1995).

governing the system is given as

$$\frac{dq}{dt} + \theta q = -\, r(q)$$

or
$$\frac{dq}{dt} = -(\theta q + r) \tag{4.1}$$

Integrating the above expression, using the conditions $q = S$ at $t = 0$ and $q = 0$ at $t = T$, the length T of each cycle is given by

$$T = \int_0^S \frac{dq}{\theta q + r} = F(S) \tag{4.2}$$

where S is the highest stock level. If C_1 be the unit holding cost per unit time then the total holding cost during time T is given by $C_1G(S)$.

where
$$G(S) = \int_0^S \frac{q}{\theta q + r}\, dq \tag{4.3}$$

Now the total profit per unit time during time T is

$$Z(S) = \frac{p(S - \theta ST) - \{A + CS + C_1G(S)\}}{F(S)} \tag{4.4}$$

Here A is the set up cost for each cycle, p is the unit selling price of the item and C is the unit cost price of the item. The optimum value of S for maximum total profit per unit time is a solution of $Z'(S) = 0$ for which $Z''(S) < 0$. Thus for optimum value of S, expression (4.4) provides,

$$F(S)\left[p\{1 - \theta F(S)\} - C - C_1G'(S)\right] = F'(S)\left[(p - C)S - A - C_1G(S)\right] \tag{4.5}$$

where prime denotes derivative with respect to S. Equation (4.5) is in general a non-linear equation which can be solved numerically by Newton-Raphson method if the explicit form of r(q) is known. The optimum cycle length is $F(S^*)$ where S^* is the optimum value of S.

ANALYSIS

EOQ model has been established for the following functional relations

(i) $r(q) = \alpha + \beta q$

(ii) $r(q) = \dfrac{\alpha}{q}$

(iii) $r(q) = \alpha + \beta q + \gamma q^2$

where α, β and γ are non-negative constants.

Case (i) : $r(q) = \alpha + \beta q$

Substituting this value of r in equation (4.2), we get

$$F(S) = \int_0^S \frac{dq}{\alpha + (\beta + \theta)q}$$

$$= \frac{1}{(\beta + \theta)} \log \left[\frac{\alpha + (\beta + \theta)S}{\alpha}\right] \tag{4.6}$$

From equation (4.3), we have,

$$G(S) = \int_0^S \frac{q\,dq}{\alpha + (\beta + \theta)q}$$

$$= \frac{1}{(\beta + \theta)} \int_0^S \left[1 - \frac{\alpha}{\alpha + (\beta + \theta)q}\right] dq$$

$$= \frac{1}{(\beta + \theta)}\left[S - \frac{\alpha}{(\beta + \theta)} \log \left\{\frac{\alpha + (\beta + \theta)S}{\alpha}\right\}\right] \tag{4.7}$$

Differentiating equation (4.6) and (4.7) with respect to S, we get,

$$F'(S) = \frac{1}{\alpha + (\beta + \theta)S} \tag{4.8}$$

and
$$G'(S) = \frac{S}{\alpha + (\beta + \theta)S} \tag{4.9}$$

Putting these values of F(S), G(S), $F'(S)$ and $G'(S)$ in optimal condition (4.5) and simplifying the resulting expression, we get

$$A - \left\{C_1 - (\beta+\theta)(p - C)\right\} \frac{\left[\left\{\alpha + (\beta+\theta)S\right\} \log\left\{\frac{\alpha + (\beta+\theta)S}{\alpha}\right\} - (\beta+\theta)S\right]}{(\beta+\theta)^2}$$

$$- \frac{p\theta}{(\beta+\theta)^2}\left[\log \frac{\left\{\alpha + (\beta+\theta)S\right\}}{\alpha}\right]^2 = 0 \qquad (4.10)$$

If we take $\theta = 0$ i.e., there be no deterioration expression (4.10) reduces to the expression obtained by Mandal and Phaujdar [72].

Equation (4.10) is a transcedental equation which can be solved numerically by Newton Raphson method. It can be shown that $Z''(S) < 0$ for this value of S. Thus the solution of equation (4.10) is optimal value of S.

Case (ii) : $r(q) = \frac{\alpha}{q}$

Putting this value of r in equations (4.2) and (4.3) respectively, we have

$$F(S) = \int_0^S \frac{q\,dq}{\theta q^2 + \alpha}$$

$$= \frac{1}{2\theta} \log\left[\frac{\theta S^2 + \alpha}{\alpha}\right] \qquad (4.11)$$

and $$G(S) = \int_0^S \frac{q^2\,dq}{\theta q^2 + \alpha} = \frac{1}{\theta}\int_0^S \left[1 - \frac{\alpha}{\theta q^2 + \alpha}\right] dq$$

$$= \frac{1}{\theta}\left[S - \sqrt{\frac{\alpha}{\theta}} \tan^{-1} \sqrt{\frac{\alpha}{\theta}}\, S\right] \qquad (4.12)$$

Differentiating equations (4.11) and (4.12) with respect to S, we get

$$F'(S) = \frac{S}{\theta S^2 + \alpha} \qquad (4.13)$$

and $$G'(S) = \frac{S^2}{\theta S^2 + \alpha} \qquad (4.14)$$

Now substituting these values of F(S), G(S), F'(S) and G'(S) in equation (4.5) and simplifying the resulting expression, we have

$$A + \frac{C_1}{\theta}\left[S - \sqrt{\frac{\alpha}{\theta}}\tan^{-1}\sqrt{\frac{\alpha}{\theta}}\,S - \frac{S}{2}\log\left(\frac{\theta S^2+\alpha}{\alpha}\right)\right] - \frac{(p-C)}{2\theta S}\left[2\theta S^2\right.$$

$$\left. - (\theta S^2+\alpha)\log\left(\frac{\theta S^2+\alpha}{\alpha}\right)\right] - \frac{p(\theta S^2+\alpha)}{4\theta S}\left[\log\left(\frac{\theta S^2+\alpha}{\alpha}\right)\right]^2 = 0 \qquad (4.15)$$

Equation (4.15) is a transcedental equation which can be solved numerically by Newton Raphson method. If this solution satisfies $Z''(S) < 0$ then the solution is a optimum value of S.

Case (iii) $r(q) = \alpha + \beta q + \gamma q^2$

Substituting the value of r in equation (4.2) we get,

$$F(S) = \int_0^S \frac{dq}{\alpha + (\beta+\theta)\,q + \gamma q^2}$$

$$= \frac{1}{N}\log\left[\frac{2S\gamma + (\beta+\theta) - N}{2S\gamma + (\beta+\theta) + N}\right]\left[\frac{(\beta+\theta)+N}{(\beta+\theta)-N}\right] \qquad (4.16)$$

where $N^2 = [\,(\beta+\theta)^2 - 4\alpha\gamma] \;>\; 0$

Therefore

$$F'(S) = \frac{4\gamma}{\left[\left\{2S\gamma + (\beta+\theta)\right\}^2 - N^2\right]} \qquad (4.17)$$

Also from equation (4.3), we have

$$G(S) = \int_0^S \frac{q\,dq}{\alpha + (\beta+\theta)q + \gamma q^2}$$

$$= \frac{1}{2\gamma}\left[\frac{\alpha + (\beta+\theta)S + \gamma S^2}{\alpha}\right] - \frac{(\beta+\theta)}{2\gamma N}\log\left[\frac{2S\gamma + (\beta+\theta) - N}{2S\gamma + (\beta+\theta) + N}\right]\left[\frac{\beta+N}{\beta-N}\right] \qquad (4.18)$$

So $$G'(S) = \frac{(\beta+\theta) + 2\gamma S}{2\gamma + [\alpha + (\beta+\theta)S + \gamma S^2]} - \frac{2(\beta+\theta)}{\left[\left\{2\gamma S + (\beta+\theta)\right\}^2 - N^2\right]} \qquad (4.19)$$

Now substituting the values of F(S), F'(S), G(S) and G'(S) from

equations (4.16), (4.17), (4.18) and (4.19) respectively in equation (4.5) and simplifying the resulting expression, we get

$$A + \frac{C_1}{2\gamma}\log\left[\frac{\alpha + (\beta+\theta)S + \gamma S}{\alpha}\right]^2 - \frac{1}{4\gamma N}\log\frac{\left[1 + \frac{2S\gamma}{(\beta+\theta) - N}\right]}{\left[1 + \frac{2S\gamma}{(\beta+\theta) + N}\right]}$$

$$\times\left[(p-C)\left[\left\{(\beta+\theta)+2\gamma S\right\}^2 - N^2\right] - \frac{C_1[(\beta+\theta)+2\gamma S][\{(\beta+\theta)+2\gamma S\}^2 - N^2]}{2\gamma[\alpha+(\beta+\theta)S+\gamma S^2]}\right]$$

$$+ 2C_1(\beta+\theta) = 0 \qquad (4.20)$$

In particular if $\gamma = 0$, the above expression (4.20) reduces to equation (4.10) of case (i), and if also $\theta = 0$, then this reduces to that derived by **Mandal and Phaujdar** [72].

Equation (4.20) is a transcedental equation which can be solved by Newton Raphson method. The solution $S = S^*$ of equation (4.20) such that $Z''(S^*) < 0$, gives the optimal value of S.

DISCUSSION

In this chapter an EOQ model is derived in which production occurs instantaneously and items in inventory deteriorate at a constant rate. Some functional relationships existing between the demand rate and current stock level are considered to establish the inventory model. It is also shown that on restricting the conditions on demand rates the above model reduces to that given by **Mandal and Phaujdar** [72].

This EOQ model can further be generalised by taking some more suitable dependence of stock quantity on demand rate.

CHAPTER – 5

AN ORDER-LEVEL INVENTORY MODEL FOR DETERIORATING ITEMS WITH FINITE RATE OF REPLENISHMENT

INTRODUCTION

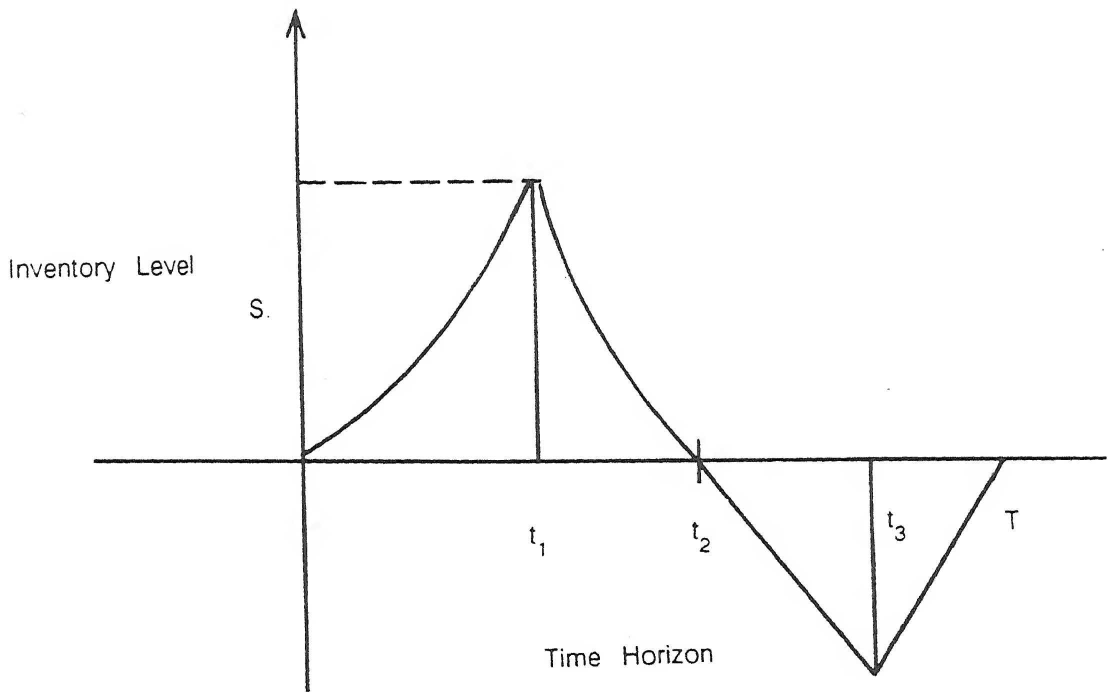
Shah and Jaiswal [91] developed an order-level inventory model for the items with constant rate of deterioration. But they made a wrong assumption regarding the inventory holding cost. Actually the authors have considered the inventory level of the system exactly linear in the prescribed interval which is incorrect. **Aggarwal, S.P.** [1] developed an order-level inventory model of rectifying the error in the analysis of **Shah and Jaiswal** [91] in calculating the average inventory holding cost in the model. Then after an order-level inventory system with power demand was formulated by **Goel and Aggarwal** [47]. In all these inventory model production is instantaneous. **Misra** [67] developed a deterministic model with finite rate of replenishment and without shortages. He considered both constant and variable rates of deterioration.

Mitali Roy Chowdhury and K.S. Chowdhuri [19] derived an order-level inventory model for constantly deteriorating items and finite rate of replenishment. There are some mistakes in their calculation of the value of time t_1 at which the production stops and attains highest level S and therefore the average inventory holding cost is also incorrect.

In this chapter an order-level inventory model with constant rate of deterioration and finite rate of replenishment, taking shortages into account

---

A paper on the work of this chapter has been communicated for publication in 'Opsearch'.



Graphical representation of Order Level Inventory Model

is presented by modifying the errors occured in the analysis of **Mitali Roy Chowdhury and K.S. Chaudhuri** [19]. Both the deterministic as well as probabilistic models are discussed.

THE MODEL

A deterministic order-level model for deteriorating item with finite rate of replenishment is developed under following assumptions–

(i) The demand rate is r.

(ii) The lead time is zero.

(iii) T is the fixed duration of each production cycle.

(iv) The production rate k is finite and $k > r$.

(v) Shortages are allowed and backlogged.

(vi) A constant fraction θ of the on hand inventory deteriorates per unit time.

The amount of stock is zero at $t = 0$. Production starts at $t = 0$ and stops at $t = t_1$ attaining a level S. During (t_1, t_2) the inventory level gradually decreases mainly to meet up demands and partly for deterioration. By this process the stock reaches zero level at $t = t_2$. Now shortages occur and accumulate to the level P at $t = t_3$. Production starts again at $t = t_3$ and backlog is cleared at $t = T$. The cycle then repeats itself after time T. A graphical representation of the production cycle is shown on the adjacent page.

ANALYSIS

DETERMINISTIC DEMAND CASE

Let q(t) denotes the inventory level of the syatem at time t $(0 \le t \le T)$, then the differential equations describing the instantaneous states of q(t) in the interval (0, T) are

$$\frac{d}{dt}q(t) + \theta\, q(t) = k - r, \qquad 0 \le t \le t_1 \tag{5.1}$$

$$\frac{d}{dt}q(t) + \theta\, q(t) = -r, \qquad t_1 < t \le t_2 \tag{5.2}$$

$$\frac{d}{dt}q(t) = -r, \qquad t_2 < t \le t_3 \tag{5.3}$$

$$\frac{d}{dt}q(t) = k - r, \qquad t_3 < t \le T \tag{5.4}$$

Solutions to these are

$$q(t) = \frac{k-r}{\theta}\left(1 - e^{-\theta t}\right), \qquad 0 \le t \le t_1 \tag{5.5}$$

$$= \left(S + \frac{r}{\theta}\right) e^{-\theta(t - t_1)} - \frac{r}{\theta}, \qquad t_1 < t \le t_2 \tag{5.6}$$

$$= r(t_2 - t), \qquad t_2 < t \le t_3 \tag{5.7}$$

$$= (k-r)(t - t_3) - P, \qquad t_3 < t \le T \tag{5.8}$$

From equation (5.5), using $q(t_1) = S$, we have

$$S = \frac{k-r}{\theta}\left(1 - e^{-\theta t_1}\right)$$

or $$e^{-\theta t_1} = \left(1 - \frac{S\theta}{k-r}\right)$$

or $$t_1 = -\frac{1}{\theta}\log\left(1 - \frac{S\theta}{k-r}\right) \tag{5.9}$$

or $$\theta t_1 = \left[\frac{\theta S}{k-r} + \frac{\theta^2 S^2}{2(k-r)^2} + \frac{\theta^3 S^3}{3(k-r)^3} + \;.\;.\;.\;.\right] \tag{5.10}$$

From equation (5.6) using $q(t_2) = 0$, we have

$$0 = \left(S + \frac{r}{\theta}\right) e^{-\theta(t_2 - t_1)} - \frac{r}{\theta}$$

or $$e^{\theta(t_2 - t_1)} = \left(1 + \frac{\theta S}{r}\right) \tag{5.11}$$

or $$t_2 - t_1 = \frac{1}{\theta}\log\left(1 + \frac{\theta S}{r}\right)$$

$$= \frac{1}{\theta}\left[\frac{\theta S}{r} - \frac{\theta^2 S^2}{2r^2} + \frac{\theta^3 S^3}{3r^3} - \quad . \quad . \quad . \quad . \right]$$

$$= \left[\frac{S}{r} - \frac{\theta S^2}{2r^2} + \frac{\theta^2 S^3}{3r^3} - \quad . \quad . \quad . \quad . \right] \tag{5.12}$$

Also from equation (5.10),

$$t_2 = t_1 + \frac{1}{\theta}\log\left(1 + \frac{\theta S}{r}\right)$$

or $$t_2 = -\frac{1}{\theta}\log\left(1 - \frac{\theta S}{k - r}\right) + \frac{1}{\theta}\log\left(1 + \frac{\theta S}{r}\right)$$

$$= \frac{1}{\theta}\log\left(\frac{k - r}{k - r - \theta S}\right) + \frac{1}{\theta}\log\left(1 + \frac{\theta S}{r}\right)$$

$$= \frac{1}{\theta}\log\left[\left(\frac{k - r}{k - r - \theta S}\right) \times \left(\frac{r + \theta S}{r}\right)\right]$$

$$= \frac{1}{\theta}\log\left(\frac{kr - r^2 - \theta Sr + k\theta S}{kr - r^2 - \theta Sr}\right)$$

$$= \frac{1}{\theta}\log\left[1 + \frac{\theta kS}{r(k - r - \theta S)}\right]$$

$$= \frac{1}{\theta}\log\left[1 + \frac{\theta kS}{r(k - r)}\right] \qquad \text{as} \qquad 0 < \theta << 1 \tag{5.13}$$

From equations (5.7) and (5.8) using $q(t_3) = P$ and $q(T) = 0$ respectively, we get

$$S(t_3 - t_2) = P \tag{5.14}$$

and $$(k - r)(T - t_3) = P \tag{5.15}$$

Now the total number of units in inventory

$$= \frac{k - r}{\theta}\int_0^{t_1}\left(1 - e^{-\theta t}\right) dt + \int_{t_1}^{t_2}\left[\left(S + \frac{r}{\theta}\right) e^{-\theta(t - t_1)} - \frac{r}{\theta}\right] dt$$

$$= \frac{k-r}{\theta}\left[t + \frac{e^{-\theta t}}{\theta}\right]_0^{t_1} - \frac{1}{\theta}\left(S + \frac{r}{\theta}\right)\left[e^{-\theta(t-t_1)}\right]_{t_1}^{t_2} - \frac{r}{\theta}(t_2 - t_1)$$

$$= \frac{k-r}{\theta^2}[\theta t_1 + e^{-\theta t_1} - 1] - \left(\frac{S}{\theta} + \frac{r}{\theta^2}\right)\left[e^{-\theta(t_2 - t_1)} - 1\right] - \frac{r}{\theta}(t_2 - t_1)$$

$$= \frac{k-r}{\theta^2}\left(\frac{\theta^2 t_1^2}{2} - \frac{\theta^3 t_1^3}{6} + \quad . \quad . \quad .\right) - \left(\frac{S}{\theta} + \frac{r}{\theta^2}\right)\left[-\theta(t_2 - t_1) + \frac{\theta^2}{2}(t_2 - t_1)^2\right.$$

$$\left. - \frac{\theta^3}{6}(t_2 - t_1)^3 + \quad . \quad . \quad .\right] - \frac{r}{\theta}(t_2 - t_1)$$

$$= \frac{k-r}{\theta^2}\left[\frac{\theta^2 S^2}{2(k-r)^2} + \frac{\theta^3 S^3}{2(k-r)^3} - \frac{\theta^3 S^3}{6(k-r)^3} + \quad . \quad . \quad .\right] - S(t_2 - t_1)$$

$$- \frac{S\theta}{2}(t_2 - t_1)^2 - \frac{r}{2}(t_2 - t_1)^2 + \frac{r\theta}{6}(t_2 - t_1)^3 \; ; \qquad 0 < \theta << 1$$

$$= \left[\frac{S^2}{2(k-r)^2} + \frac{\theta S^3}{3(k-r)^2}\right] + S\left(\frac{S}{r} - \frac{\theta S^2}{2r^2}\right) - \frac{\theta S}{2}\left(\frac{S^2}{r^2}\right)$$

$$- \frac{r}{2}\left(\frac{S^2}{r^2} - \frac{\theta S^3}{r^3}\right) + \frac{r\theta}{6}\left(\frac{S^3}{r^3}\right)$$

$$= \frac{S^2}{2(k-r)} + \frac{\theta S^3}{3(k-r)^2} + \frac{S^2}{r} - \frac{\theta S^3}{2r^2} - \frac{\theta S^3}{2r^2} - \frac{S^2}{2r} + \frac{\theta S^3}{2r^2} + \frac{\theta S^3}{6r^2}$$

$$= \frac{S^2}{2(k-r)} + \frac{S^2}{2r} + \frac{\theta S^3}{3(k-r)^2} - \frac{\theta S^3}{3r^2}$$

$$= \frac{S^2 k}{2r(k-r)} - \frac{\theta S^3 k(k-2r)}{3r^2(k-r)^2} \tag{5.16}$$

Also the total number of deteriorated items during (0, T) is

$$D = [(k-r)t_1 - S] + [S - r(t_2 - t_1)]$$

$$= (k-r)t_1 - r(t_2 - t_1)$$

$$= (k-r)\left[\frac{S}{(k-r)} + \frac{\theta S^2}{2(k-r)^2}\right] - r\left[\frac{S}{r} - \frac{\theta S^2}{2r^2}\right]$$, from equations (5.10) and (5.12) respectively.

$$= \frac{\theta S^2}{2}\left[\frac{1}{k-r} + \frac{1}{r}\right]$$

$$= \frac{k\theta S^2}{2r(k-r)} \tag{5.17}$$

And total number of units responsible for shortages are,

$$r\int_{t_2}^{t_3} (t_2 - t_1)\,dt + \int_{t_3}^{T} \left[(k-r)(t-t_3) - P\right] dt$$

$$= -\frac{r}{2}\left[(t_2 - t_1)^2\right]_{t_2}^{t_3} + \frac{(k-r)}{2}\left[(t-t_3)^2\right]_{t_3}^{T} - P\left(t\right)_{t_3}^{T}$$

$$= -\frac{r}{2}(t_2 - t_3)^2 + \frac{(k-r)}{2}(T - t_3)^2 - P(T - t_3)$$

$$= -\frac{P^2}{2r} + \frac{P^2}{2(k-r)} - \frac{P^2}{(k-r)}$$ using equations (5.14) and (5.15).

$$= -\frac{P^2}{2}\left[\frac{1}{r} + \frac{1}{k-r}\right]$$

$$= -\frac{P^2 k}{2r(k-r)} \tag{5.18}$$

Here negative sign indicates shortages.

Therefore total average cost of the system

$$K(S,\ P) = \frac{C_1 S^2 k}{2r(k-r)T} - \frac{C_1 \theta S^3 k(k-2r)}{3r^2(k-r)^2 T} + \frac{C_2 P^2 k}{2r(k-r)T} + \frac{Ck\theta S^2}{2r(k-r)T} \tag{5.19}$$

Where C_1 is inventory holding cost per unit per unit time, C_2 is shortage cost per unit per unit time and C the cost of each unit.

Since S and P are dependent variables, hence necessary condition

for K to be minimum is

$$\frac{dK}{dS}(S, P) = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial P} \cdot \frac{\partial P}{\partial S} = 0 \tag{5.20}$$

Substituting values of t_2 and t_3 from equations (5.13) and (5.15) respectively in equation (5.14), we have

$$r\left[T - \frac{P}{k-r} - \frac{Sk}{r(k-r)} + \frac{\theta S^2k^2}{2r^2(k-r)^2}\right] = P, \qquad 0 < \theta << 1$$

or $$\frac{Pk}{r(k-r)} = rT - \frac{Sk}{r(k-r)} + \frac{\theta S^2k^2}{2r^2(k-r)^2}$$

or $$P = \frac{r(k-r)T}{k} - S + \frac{\theta S^2k}{2r(k-r)} \tag{5.21}$$

Differentiating equation (5.21) with respect to S, this gives,

$$\frac{dP}{dS} = \frac{\theta Sk}{r(k-r)} - 1 \tag{5.22}$$

Using equations (5.19) and (5.22) in optimal condition (5.20), we get

$$\left[C_1S - \frac{C_1(k-2r)\theta S^2}{r(k-r)} + C\theta S\right] + C_2P\left[\frac{\theta Sk}{r(k-r)} - 1\right] = 0$$

or $$\left[C_1S - \frac{C_1(k-2r)\theta S^2}{r(k-r)} + C\theta S\right] + C_2\left[\frac{Tr(k-r)}{k} - S + \frac{\theta S^2k}{2r(k-r)}\right]\left[\frac{\theta Sk}{r(k-r)} - 1\right] = 0$$

or $$\left[C_1S - \frac{C_1(k-2r)\theta S^2}{r(k-r)} + C\theta S\right] + C_2\left[\theta ST + S - \frac{r(k-r)T}{k} - \frac{3}{2}\frac{\theta S^2k}{r(k-r)}\right] = 0$$

or $$\left[\frac{(k-2r)\theta}{r(k-r)}C_1 + \frac{3k\theta}{2r(k-r)}C_2\right]S^2 - \left[C\theta + C_1 + (1+T\theta)C_2\right]S + \frac{r(k-r)T}{k}C_2 = 0 \tag{5.23}$$

This equation is a quadratic equation in S and its solution gives optimum value S^* of S if $\frac{d^2}{dS^2}K(S^*) > 0$.

Numerical Example :

An item is produced at a rate of 250 per year and the demand occurs at a rate of 100 per year. If the cost of each item is Rs. 0.20, holding cost is Rs. 0.30 per unit per year, shortage cost is Rs. 1.50 and the rate of deterioration is 0.01 per year, determine the optimal quantities.

Here C_1 = Rs. 0.30 per unit per year

C_2 = Rs. 1.50 per unit per year

C = Rs. 0.20 per unit

$\theta = 0.01$ units per year

$k = 250$ units per year

$r = 100$ units per year

Substituting these values in equation (5.23), we get

$$2.88750\, S^2 - 3.02833 \times 10^{-5}\, S + 6.75 \times 10^5 = 0$$

which is a quadratic equation in S and has both the roots real and positive. Only one root $S = 50.1$ is admissible as other root does not satisfy the condition for minimum cost.

So $S = 50.1$ is the optimum order level.

Using equation (5.21), we have $P = 9.7$

From equation (5.9) and (5.15),

$t_1 = 0.33$ year and $T - t_3 = 0.06$

Finally, from equation (5.19) minimum cost

$K(S, P)$ = Rs. 7.49 per year.

PROBABILISTIC DEMAND CASE

Let f(x) be the probability density function for demand x which is a random variable during scheduling period T $(0 \le x < \infty)$ and is uniform.

Also at the beginning of each period the initial stock status is zero.

Case I—When shortages do not occur.

Let $q_x(t)$ be the inventory level at any time t $(0 \le t \le T)$ of the system and x is the demand during the fixed interval T then the demand rate is $\frac{x}{T}$ during the period.

The differential equations governing the system are

$$\frac{d}{dt} q_x(t) + \theta q_x(t) = k - \frac{x}{T}, \qquad 0 \le t \le t_1 \tag{5.24}$$

and $$\frac{d}{dt} q_x(t) + \theta q_x(t) = -\frac{x}{T}, \qquad t_1 < t \le T \tag{5.25}$$

Solution to which using conditions

$q_x(0) = 0$, $q_x(t_1) = S$, $q_x(T) = 0$ are given by

$$q_x(t) = \frac{k - \frac{x}{T}}{\theta}\left(1 - e^{-\theta t}\right), \qquad 0 \le t \le t_1 \tag{5.26}$$

$$= \left(S + \frac{x}{\theta T}\right) e^{-\theta(t - t_1)} - \frac{x}{\theta T}, \qquad t_1 < t \le T \tag{5.27}$$

As shortages do not occur, we have

$q_x(T) \ge 0$, which using equation (5.27) gives

$$(S\theta T + x)\, e^{-\theta(T - t_1)} \ge x$$

or $$x \le \frac{S\theta T}{e^{\theta(T - t_1)} - 1} = S^* \tag{5.28}$$

Therefore the expected average inventory is

$$H_1(x) = \frac{1}{T}\int_0^T q_x(t)\, dt, \qquad x \le S^*$$

$$= \frac{kt_1}{\theta T} - \frac{1}{\theta T}\left(S + \frac{x}{\theta T}\right) e^{-\theta(T - t_1)} + \frac{x}{\theta^2 T^2}(1 - \theta T) \tag{5.29}$$

where $t_1 = -\frac{1}{\theta}\log\left(1 - \frac{\theta S}{k - x/T}\right)$, $\qquad x \le S^*$ (5.30)

and expected average shortage during this period is

$$G_1(x) = 0 \tag{5.31}$$

Also the expected number of deteriorated items is

$$D_1(x) = kt_1 - x - q_x(t) \tag{5.32}$$

Case II : When shortages occur

In this situation, the system is the same as that of the deterministic model except that the demand x is a random variable with probability density function f(x) during the scheduling period T. Therefore solutions are obtained, replacing r by $\frac{x}{T}$ in equations (5.5), (5.6), (5.7) and (5.8).

$$q_x(t) = \frac{k - x/T}{\theta}\left(1 - e^{-\theta t}\right), \qquad 0 \le t \le t_1 \tag{5.33}$$

$$= \left(S + \frac{x}{\theta T}\right) e^{-\theta(t - t_1)} - \frac{x}{\theta T}, \qquad t_1 < t \le t_2 \tag{5.34}$$

$$= \frac{x}{T}(t_2 - t_1), \qquad t_2 < t \le t_3 \tag{5.35}$$

$$= \left(k - \frac{x}{T}\right)(t - t_3) - P, \qquad t_3 < t \le T \tag{5.36}$$

Since shortages occur, we have

$q_x(T) < 0$ which gives using equation (5.36)

or $\qquad x > kT - \frac{P}{1 - t_3/T} = P^*$ (5.37)

Therefore the expected average inventory is

$$H_2(x) = \frac{1}{T}\int_0^{t_2} q_x(t)\,dt\,, \qquad x > P^*$$

$$= \frac{1}{T}\left[\frac{S^2k}{s\left(\frac{x}{T}\right)\left(k-\frac{x}{T}\right)} - \frac{\theta S^3 k\left(k-\frac{2x}{T}\right)}{\left(\frac{3x^2}{T^2}\right)\left(k-\frac{x}{T}\right)^2}\right], \qquad x > P^* \qquad (5.38)$$

and the expected average shortages during the period,

$$G_2(x) = \frac{1}{T}\left[\frac{P^2k}{\left(\frac{2x}{T}\right)\left(k-\frac{x}{T}\right)}\right], \qquad x > P^* \qquad (5.39)$$

where $$t_2 = \frac{1}{\theta}\log\left[1+\frac{\theta S k}{\frac{x}{T}\left(k-\frac{x}{T}\right)}\right] \qquad (5.40)$$

$$P = \frac{x}{T}(t_3 - t_2) \qquad (5.41)$$

and $$\left(k-\frac{x}{T}\right)(T-t_3) = \alpha \quad \text{(Negative constant)} \qquad (5.42)$$

Finally the expected number of items which deteriorate during the fixed interval is

$$D_2(x) = \left[\left(k-\frac{x}{T}\right)t_1 - S\right] + \left[S - \frac{x}{T}(t_2 - t_1)\right]$$

$$= \frac{k\theta S^2}{2\left(\frac{x}{T}\right)\left(k-\frac{x}{T}\right)} \qquad (5.43)$$

Therefore, the expected total cost of the system is

$$K(S,\ P) = C_1\left[\int_0^{S^*} H_1(x)\,f(x)\,dx + \int_{P^*}^{\infty} H_2(x)\,f(x)\,dx\right] + C_2\left[\int_{P^*}^{\infty} G_2(x)\,f(x)\,dx\right]$$

$$+ \frac{C}{T} \left[\int_{0}^{S^*} D_1(x)\, f(x)\, dx + \int_{P^*}^{\infty} D_2(x)\, f(x)\, dx \right] \tag{5.44}$$

From equations (5.41) and (5.42) eliminating t_3, a relation between P and S is obtained. Using this relation and condition (5.20) for optimality of S, optimum value of S can be obtained as in case of deterministic model.

DISCUSSION

An order level inventory model for deteriotating items has beer studied. Here production occurs at a constant rate k, items consume at a rate r (k > r) and shortages are allowed for a fixed time period, which are completely backlogged. The model is derived for both the cases o demand, i.e. deterministic as well as probabilistic.

The above model can be further generalized for variable rate o deterioration of items. Demand rate can also be taken in some othe functional form.

Chapter – 6

INVENTORY MODELS WITH VARIABLE DEMAND

INTRODUCTION

We have seen that classical EOQ model [55, 75] does not have involvement of stock level q or time t on demand rate. So many authors have produced in past, several inventory models, taking the dependence of stock level or time on demand rate. **Mandal and Phaujdar** [72, 73], **Gupta and Vrat** [50] considered the stock dependent demand rate in their EOQ models. **Silver and Meal** [90] developed an approximate solution procedure for the general case of a deterministic, time-varying demand pattern. The classical no-shortage inventory problem for a linear trend in demand over a finite time-horizon was analytically solved by **Donaldson** [23]. **Silver** [95] derived a heuristic for the special case of a positive, linear trend in demand and applied it to the problem of Donaldson to cut short his complicated computational procedure. **Ritchie** [85] obtained an exact solution, having the simplicity of the EOQ formula, for Donaldson problem for linear, increasing demand. **Deb and Chaudhari** [30] studied the inventory replenishment policy for items having a deterministic demand pattern with a linear trend and shortages. They developed a heuristic to determine the decision rule and sizes of replenishments over a finite time horizon to keep total costs minimum. Further this work was extended by **Murdeshwar** [71].

In all the above inventory models, demand is either function of stock level or time. But in some cases demand may depend on both stock-level as well as time. For example, products showing a seasonal trend.

The present chapter deals with those inventory models which take care of demand rate on stock level and time both. Replenishment is

---

A paper on the work of this chapter has been communicated in Opsearch.

instantaneous and shortages are not permitted.

ANALYSIS

In the first section profit maximization technique is used to estimate total average profit per unit time, while in the second section cost minimization technique is applied to determine minimum average cost per unit time of the system respectively.

SECTION – 1

PROFIT MAXIMIZATION CRITERIA

The EOQ model has been derived for the following functionals of demand rate and all the assumptions are same as in the inventory model of third chapter.

(i) $r(q) = t\,e^{\alpha q}$

(ii) $r(q) = \dfrac{\alpha + \beta q}{\alpha + \beta t}$, where α, β are constants and q is the stock level at any time t.

Case (i) : Let $r(q) = t\,e^{\alpha q}$, then

$$\frac{dq}{dt} = -\,t\,e^{\alpha q} \tag{6.1}$$

Now considering $q(0) = S$ and $q(T) = 0$, the length T of each cycle is calculated using equation (6.1) as,

$$\int_0^T t\,dt = -\int_0^S \frac{dq}{e^{\alpha q}}$$

or $$\frac{T^2}{2} = \int_0^S \frac{dq}{e^{\alpha q}} = -\frac{1}{\alpha}\Big[e^{-\alpha q}\Big]_0^S$$

or $$\frac{T^2}{2} = \frac{1}{\alpha}(1 - e^{-\alpha S})$$

or $$T = \frac{\sqrt{2}}{\sqrt{\alpha}}\sqrt{1 - e^{-\alpha S}} = F(S) \quad \text{Say,} \tag{6.2}$$

Differentiating expression (6.2) with respect to S, we get,

$$F'(S) = \frac{\sqrt{\alpha}}{\sqrt{2}} \frac{e^{-\alpha S}}{\sqrt{1 - e^{-\alpha S}}} \tag{6.3}$$

Also from (6.1), on integrating we have,

$$-\frac{e^{-\alpha q}}{\alpha} = -\frac{t^2}{2} + B, \text{ B is constant of integration} \tag{6.4}$$

Applying $q(0) = S$

$$B = -\frac{e^{-\alpha S}}{2}$$

So equation (6.4) reduces to

$$e^{\alpha q} = \frac{\alpha}{2} t^2 + e^{-\alpha S}$$

or $$q = \frac{1}{\alpha} \log \left(\frac{\alpha}{2} t^2 + e^{-\alpha S} \right) \tag{6.5}$$

Hence total amount of inventory during the cycle (0, T) is

$$G(S) = \int_0^T q \, dt$$

$$= -\frac{1}{\alpha} \int_0^T \log \left(\frac{\alpha t^2}{2} + e^{-\alpha S} \right) dt$$

$$= -\frac{1}{\alpha} \left[t \log \left(\frac{\alpha t^2}{2} + e^{-\alpha S} \right) \right]_0^T + \frac{1}{\alpha} \int_0^T \frac{\alpha t^2}{\left(\frac{\alpha t^2}{2} + e^{-\alpha S} \right)} dt$$

$$= -\frac{T}{\alpha} \log \left(\frac{\alpha T^2}{2} + e^{-\alpha S} \right) + \frac{2}{\alpha} \int_0^T dt - \frac{2}{\alpha} e^{-\alpha S} \int_0^T \frac{dt}{\left(\frac{\alpha t^2}{2} + e^{-\alpha S} \right)}$$

$$= -\frac{T}{\alpha} \log \left(\frac{\alpha T^2}{2} + e^{-\alpha S} \right) + \frac{2T}{\alpha} - \frac{4}{\alpha^2} e^{-\alpha S} \int_0^T \frac{dt}{t^2 + \left(\frac{\sqrt{2}}{\sqrt{\alpha}} e^{-\alpha S} \right)^2}$$

$$= -\frac{T}{\alpha}\log\left(\frac{\alpha T^2}{2} + e^{-\alpha S}\right) + \frac{2T}{\alpha} - \frac{2\sqrt{2}}{\alpha\sqrt{\alpha}}\sqrt{e^{-\alpha S}}\tan^{-1}\left(\frac{T\sqrt{\alpha}}{\sqrt{2}\sqrt{e^{-\alpha S}}}\right)$$

(6.6)

Substituting the value of T from equation (6.2) in above equation (6.6), this reduces to,

$$G(S) = \frac{2\sqrt{2}}{\alpha\sqrt{\alpha}}\sqrt{1-e^{-\alpha S}} - \frac{2\sqrt{2}}{\alpha\sqrt{\alpha}}\sqrt{e^{-\alpha S}}\tan^{-1}\left(\frac{\sqrt{1-e^{-\alpha S}}}{\sqrt{e^{-\alpha S}}}\right)$$

$$= \frac{2\sqrt{2}}{\alpha\sqrt{\alpha}\sqrt{e^{\alpha S}}}\left[\sqrt{e^{\alpha S}-1} - \tan^{-1}(\sqrt{e^{\alpha S}-1})\right] \qquad (6.7)$$

Also, differentiating equation (6.7) with respect to S, we get,

$$G'(S) = \frac{\sqrt{2}}{\sqrt{\alpha}\sqrt{e^{\alpha S}}}\tan^{-1}(\sqrt{e^{\alpha S}-1}) \qquad (6.8)$$

Finally for the total profit to be maximum, optimal condition (3.4) of chapter (3.3), using values of F(S), F′(S), G(S) and G′(S) from equations (6.2), (6.3), (6.7) and (6.8) respectively, implies,

$$\frac{\sqrt{2}}{\sqrt{\alpha}}\frac{\sqrt{e^{\alpha S}-1}}{\sqrt{e^{\alpha S}}}\left[(p-C) - \frac{\sqrt{2}}{\sqrt{\alpha}}\frac{C_1}{\sqrt{e^{\alpha S}}}\tan^{-1}(\sqrt{e^{\alpha S}-1})\right] =$$

$$\frac{\sqrt{\alpha}}{\sqrt{2}\sqrt{e^{\alpha S}}\sqrt{e^{\alpha S}-1}}\left[(p-C)S - A - \frac{2\sqrt{2}}{\alpha\sqrt{\alpha}}\cdot\frac{C_1}{\sqrt{\alpha S}}(\sqrt{e^{\alpha S-1}} - \tan^{-1}\sqrt{e^{\alpha S-1}})\right]$$

or $\quad 2(e^{\alpha S}-1)(p-C) - \dfrac{2\sqrt{2}}{\sqrt{\alpha}}\dfrac{(e^{\alpha S}-1)}{\sqrt{e^{\alpha S}}}C_1\tan^{-1}\sqrt{e^{\alpha S}-1}$

$$= \alpha(p-C)S - \alpha A - \frac{2\sqrt{2}}{\sqrt{\alpha}}\frac{C_1}{\sqrt{e^{\alpha S}}}(\sqrt{e^{\alpha S}-1} - \tan^{-1}\sqrt{e^{\alpha S}-1})$$

or $\quad \alpha A + (p-C)(2e^{\alpha S} - \alpha S - 2) + \dfrac{2\sqrt{2}}{\sqrt{\alpha}}\dfrac{C_1}{\sqrt{e^{\alpha S}}}\left[\sqrt{e^{\alpha S}-1} - e^{\alpha S}\tan^{-1}\sqrt{e^{\alpha S}-1}\right] = 0$

(6.9)

And the condition for total average cost to be minimum is given by taking $p = 0$ in above expression,

$$\alpha A - C(2e^{\alpha S} - \alpha S - 2) + \frac{2\sqrt{2}\ C_1}{\sqrt{\alpha}\ \sqrt{e^{\alpha S}}}\left[\sqrt{e^{\alpha S}-1} - e^{\alpha S}\tan^{-1}\sqrt{e^{\alpha S}-1}\right] = 0$$

(6.10)

The transcedental equations (6.9) and (6.10) can be solved by using some suitable numerical method to get optimal value S of ordered quantity. The total profit (cost) will be maximum (minimum) if

$Z''(S) < 0$, (> 0), for that value of S obtained from equation (6.9) and (6.10) respectively, where Z(S) is as given by equation (3.1) of chapter (3.3).

Case (ii) : Let $r(q) = \dfrac{\alpha + \beta q}{\gamma + \beta t}$, then

$$\frac{dq}{dt} = -\left(\frac{\alpha + \beta q}{\gamma + \beta t}\right) \qquad (6.11)$$

or $\displaystyle\int_0^T \frac{dt}{\gamma + \beta t} = -\int_S^0 \frac{dq}{\alpha + \beta q}$, as $q(0) = S$ and $q(T) = 0$

or $\dfrac{1}{\beta}\Big[\log(\gamma + \beta t)\Big]_0^T = \dfrac{1}{\beta}\Big[\log(\alpha + \beta q)\Big]_0^S$

or $\log\left(\dfrac{\gamma + \beta T}{\gamma}\right) = \log\left(\dfrac{\alpha + \beta S}{\alpha}\right)$

or $T = F(S) = \dfrac{\gamma}{\alpha} S$ (6.12)

Differentiating above equation (6.12) with respect to S, we get,

$$F'(S) = \frac{\gamma}{\alpha} \qquad (6.13)$$

Also from equation (6.11), we have,

$$\int \frac{dq}{\alpha + \beta q} = -\int \frac{dt}{\gamma + \beta t} \qquad (6.14)$$

or $\quad \frac{1}{\beta}\log(\alpha+\beta q) = -\frac{1}{\beta}\log(\gamma+\beta t) + \frac{1}{\beta}\log B$

where $\frac{1}{\beta}\log B$ is a constant of integration.

or $\quad \log(\alpha+\beta q)(\gamma+\beta t) = \log B$

or $\quad (\alpha+\beta q)(\gamma+\beta t) = B$

Since, initially at $t = 0$, $q = S$, we get,

$B = \gamma(\alpha+\beta S)$

Putting this value of B, in equation (6.14), this implies,

$(\alpha+\beta q)(\gamma+\beta t) = \gamma(\alpha+\beta S)$

or $\quad \beta q = \dfrac{\gamma(\alpha+\beta S)}{\gamma+\beta t} - \alpha$

$\quad = \dfrac{\beta(\gamma S - \alpha t)}{\gamma+\beta t}$

or $\quad q = \left(\dfrac{\gamma S - \alpha t}{\gamma+\beta t}\right) \qquad (6.15)$

Hence the total amount of inventory during the cycle (0, T) in the system is given by,

$$G(S) = \int_0^T q\,dt$$

$$= \int_0^T \left(\frac{\gamma S - \alpha t}{\gamma+\beta t}\right) dt$$

$$= \int_0^T \frac{\gamma S}{\gamma+\beta t}\,dt - \alpha\int_0^T \frac{t}{\gamma+\beta t}\,dt$$

$$= \frac{\gamma S}{\beta}\log\left(\frac{\gamma+\beta T}{\gamma}\right) - \frac{\alpha}{\beta}\int_0^T \left(1 - \frac{\gamma}{\gamma+\beta t}\right) dt$$

$$= \frac{\gamma S}{\beta} \log \left(\frac{\gamma + \beta T}{\gamma}\right) - \frac{\alpha}{\beta} T + \frac{\alpha\gamma}{\beta^2} \log \left(\frac{\gamma + \beta T}{\gamma}\right)$$

$$= \frac{\gamma}{\beta} \left(S + \frac{\alpha}{\beta}\right) \log \left(1 + \frac{\beta}{\gamma} T\right) - \frac{\alpha}{\beta} T \qquad (6.16)$$

Expression (6.16), on putting value of T from (6.12) reduces to,

$$G(S) = \frac{\gamma}{\beta} \left(S + \frac{\alpha}{\beta}\right) \log \left(1 + \frac{\beta}{\alpha} S\right) - \frac{\gamma}{\beta} S \qquad (6.17)$$

Also, differentiating equation (6.17) with respect to S, we get

$$G'(S) = \frac{\gamma}{\beta} \log \left(1 + \frac{\beta S}{\alpha}\right) \qquad (6.18)$$

Substituting values of F(S), F′(S), G(S) and G′(S) from equations (6.12), (6.13), (6.17) and (6.18) respectively in condition (3.4) of chapter (3.3) for total profit to be maximum, this gives,

$$\frac{\gamma}{\alpha} S \left[(p - C) - \frac{\gamma}{\beta} C_1 \log \left(1 + \frac{\beta}{\alpha} S\right)\right] =$$

$$\frac{\gamma}{\alpha} \left[(p - C)S - A - \frac{\gamma}{\beta} C_1 \left(S + \frac{\alpha}{\beta}\right) \log \left(1 + \frac{\beta}{\alpha} S\right) + C_1 \frac{\gamma S}{\beta}\right]$$

or $$C_1 \frac{\gamma\alpha}{\beta^2} \log \left(1 + \frac{\beta}{\alpha} S\right) + A - C_1 \frac{\gamma S}{\beta} = 0$$

or $$A + C_1\gamma \frac{\alpha \log \left(1 + \frac{\beta}{\alpha} S\right) - \beta S}{\beta^2} = 0 \qquad (6.19)$$

Equation (6.19) is a transcedental equation and its solution can be obtained by Newton-Raphson method. The solution S* of equation (6.19) which satisfy $Z''(S^*) < 0$, gives the optimum value of S, where Z(S) is given by equation (3.1) of chapter (3.3).

Expression (6.19) can also be written as

$$A + \frac{\alpha\gamma}{\beta^2} C_1 \left[\frac{\beta S}{\alpha} - \frac{\beta^2 S^2}{2\alpha^2} - \frac{\beta^3 S^3}{3\alpha^3} \cdots \right] - \frac{\gamma}{\beta} C_1 S = 0$$

or $$A + \alpha\gamma C_1 \left[-\frac{1}{2}\frac{S^2}{\alpha^2} - \frac{\beta S^3}{3\alpha^3} \cdots \right] = 0 \tag{6.20}$$

In particular if $\beta = 0$, equation (6.20) reduces to,

$$A - \frac{\gamma C_1 S^2}{2\alpha} = 0$$

or $$S = \sqrt{\frac{2A\alpha}{C_1\gamma}} \tag{6.21}$$

This value of S is the same as obtained in classical EOQ model [55] and does not depend on cost price C and selling price p of unit item.

SECTION – 2

COST MINIMIZATION CRITERIA

In this section, some more bi-variable demand rates have been considered. Cost minimization criteria is applied to derive the total average cost per unit time of the system. Functional forms of demand rate are,

(iii) $r(q) = \alpha + \beta q^{\gamma} + \delta t^{\lambda}$

(iv) $r(q) = \alpha + \beta e^{q} + \gamma e^{t}$

(v) $r(q) = \alpha + \beta q^2 t^2$, α, β, γ, δ and λ are some constants.

Case I : Let demand rate is $r(q) = \alpha + \beta q^{\gamma} + \delta t^{\lambda}$ (6.22)

Since the total average cost per unit time

$$K = rC + \frac{A}{q} r + \frac{CC_1}{2} q$$

Supplying value of r from equation (6.22), we have,

$$K(q,\ t) = C(\alpha + \beta q^{\gamma} + \delta t^{\lambda}) + \frac{A}{q}(\alpha + \beta q^{\gamma} + \delta t^{\lambda}) + \frac{CC_1}{2}q$$

$$= \alpha C + \beta C q^{\gamma} + \delta C t^{\lambda} + \frac{\alpha A}{q} + \beta A q^{\gamma - 1} + \frac{\delta A t^{\lambda}}{q} + \frac{CC_1 q}{2} \qquad (6.23)$$

Differentiating partially w.r.t. q, we get

$$\frac{\partial K}{\partial q} = \beta r C q^{\gamma - 1} - \frac{\alpha A}{q^2} + A\beta(\gamma - 1)q^{\gamma - 2)} - \frac{A\delta t^{\lambda}}{q^2} + \frac{CC_1}{2} \qquad (6.24)$$

Differentiating once again w.r.t. q partially, we get

$$\frac{\partial^2 K}{\partial q^2} = C\beta\gamma(\gamma - 1)q^{\gamma - 2)} + \frac{2A\alpha}{q^3} + A\beta(\gamma - 1)(\gamma - 2)q^{\gamma - 3} + \frac{2A\delta t^{\lambda}}{q^3} \qquad (6.25)$$

Differentiating equation (6.23) partially w.r.t. t, this gives

$$\frac{\partial K}{\partial t} = C\lambda\delta t^{\lambda - 1} + \frac{A\lambda\delta t^{\lambda - 1}}{q} \qquad (6.26)$$

Further differntiating equation (6.26) w.r.t. t, this gives

$$\frac{\partial^2 t}{\partial t^2} = C\lambda(\lambda - 1)\delta t^{\lambda - 2} + A\frac{\delta}{q}\lambda(\lambda - 1)t^{\lambda - 2} \qquad (6.27)$$

Differentiating equation (6.24) partially w.r.t. t, we get

$$\frac{\partial^2 K}{\partial q\,\partial t} = -\frac{A\lambda\delta t^{\lambda - 1}}{q^2} \qquad (6.28)$$

For K (q, t) to be minimum, the necessary condition,

$$\left(\frac{\partial^2 K}{\partial q^2}\right)\left(\frac{\partial^2 K}{\partial t^2}\right) - \left(\frac{\partial^2 K}{\partial S\,\partial t}\right) > 0$$

or $$\left[C\beta\gamma(\gamma - 1)q^{\gamma - 2} + \frac{2A\alpha}{q^3} + A\beta(\gamma - 1)(\gamma - 2)q^{\gamma - 3} + \frac{2A\delta t^{\lambda}}{q^3}\right] \times$$

$$\left[C\delta\lambda(\lambda - 1) + \frac{A\delta\lambda(\lambda - 1)}{q}\right] > \left(\frac{A\delta\lambda}{q^2}\right)^2 t^{\lambda} \qquad (6.29)$$

or $\quad C^2\beta\delta\lambda(\lambda-1)\gamma(\gamma-1)q^{\gamma-2}+2AC\dfrac{\alpha\delta\lambda}{q^3}(\lambda-1)+AC\beta\delta\lambda(\lambda-1)\times$

$$(\gamma-1)(\gamma-2)q^{\gamma-3}+2AC\frac{\delta^2\lambda}{q^3}(\lambda-1)t^{\lambda}+A\beta C\delta\lambda(\lambda-1)\gamma(\gamma-1)q^{\gamma-3}$$

$$+2\alpha\frac{A^2\delta}{q^4}\lambda(\lambda-1)+A^2\beta\delta\lambda(\lambda-1)(\gamma-1)(\lambda-2)q^{\gamma-4}$$

$$+2A^2\frac{\delta^2\lambda(\lambda-1)}{q^4}t^{\lambda} > \frac{A^2\delta^2\lambda^2}{q^4}t^{\lambda} \qquad (6.30)$$

or $\quad \beta\delta\lambda(\lambda-1)(\gamma-1)q^{\gamma-4}\left\{\gamma C^2q^2+(\gamma-2)ACq+AC\gamma q+A^2(\gamma-2)\right\}$

$$+2A\alpha\frac{\delta\lambda(\lambda-1)}{q^4}(Cq+A)+\frac{A\delta^2\lambda}{q^4}\left\{2C(\lambda-1)q+2A(\lambda-1)-A\lambda\right\}t^{\lambda} > 0$$

$$(6.31)$$

or $\quad \beta(\lambda-1)(\gamma-1)q^{\gamma}\left\{\gamma C^2q^2+2(\gamma-1)CqA+A^2(\gamma-2)\right\}+2A\alpha(\lambda-1)(Cq+A)$

$$+t^{\lambda}A\delta\left\{2C(\lambda-1)q+A(\lambda-2)\right\} > 0 \qquad (6.32)$$

Total average cost K will be minimum if $\dfrac{\partial^2K}{\partial t^2} > 0$ at values $q = q^*$ and $t = t^*$ given by optimality condition (6.32) provided $\alpha, \beta, \delta, \gamma > 0$ and $\gamma \geq 2$.

Case II. Let $r = \alpha + \beta e^q + \gamma e^t$, so that total average cost per unit time

$$K(q,\ t) = C(\alpha+\beta e^q+\gamma e^t)+\frac{A}{q}(\alpha+\beta e^q+\gamma e^t)+\frac{CC_1q}{2} \qquad (6.33)$$

Differentiating equation (6.33) partially w.r.t. q, we get

$$\frac{\partial K}{\partial q} = \beta Ce^q+\frac{A}{q^2}(q\beta e^q-\alpha-\beta e^q-\gamma e^t)+\frac{CC_1}{2} \qquad (6.34)$$

Differentiating once again with respect to q, partially, we have,

$$\frac{\partial^2 K}{\partial q^2} = \beta Ce^q + \frac{A}{q^4}\left[\beta q^3 e^q - 2q(\beta qe^q - \alpha - \beta e^q - \gamma e^t)\right]$$

$$= \beta Ce^q + \frac{A}{q^4}(\beta q^3 e^q - 2\beta q^2 e^q + 2\beta qe^q + 2\gamma qe^t + 2\alpha q) \tag{6.35}$$

Differentiating once equation (6.33) w.r.t. t partially,

$$\frac{\partial K}{\partial t} = \gamma Ce^t + \frac{A\gamma}{q}e^t = \gamma\left(C + \frac{A}{q}\right)e^t \tag{6.36}$$

and $$\frac{\partial^2 K}{\partial t^2} = \gamma\left(C + \frac{A}{q}\right)e^t \tag{6.37}$$

Also $$\frac{\partial^2 K}{\partial q\,\partial t} = -\frac{A\gamma}{q^2}e^t \tag{6.38}$$

Now total cost to be minimum, necessary condition is,

$$\left[\beta Ce^q + \frac{A}{q^4}\left(\beta q^3 e^q - 2\beta q^2 e^q + 2\beta qe^q + 2\gamma qe^t + 2\alpha q\right)\right] \times$$

$$\left[\gamma\left(C + \frac{A}{q}\right)e^t\right] > \frac{A^2\gamma^2}{q^4}e^{2t}$$

or $$\left[\left\{\left(Cq^3 + Aq^2 - 2Aq + 2A\right)\beta e^q + 2A(2\gamma e^t + \alpha)\right\}\left\{(Cq + A)\right\}\right] > A^2\gamma e^t \tag{6.39}$$

Total cost will be minimum if $\frac{d^2K}{dt^2} > 0$ at values $q = q^*$ and $t = t^*$ obtained from equation (6.39).

Case III. Let demand rate be of the form, $r = \alpha + \beta q^2 t^2, \quad \beta \neq 0$

Total cost can be given by

$$K(q,\ t) = C(\alpha + \beta q^2 t^2) + \frac{A}{q}(\alpha + \beta q^2 t^2) + \frac{CC_1 q}{2} \tag{6.40}$$

Differentiating equation (6.40) partially w.r.t. q, successively twice,

$$\frac{\partial K}{\partial q} = 2\beta Cqt^2 - \frac{A\alpha}{q^2} + A\beta t^2 + \frac{CC_1}{2} \tag{6.41}$$

and $$\frac{\partial^2 K}{\partial q^2} = 2\beta Ct^2 + \frac{2A\alpha}{q^3} \tag{6.42}$$

Differentiating equation (6.40) w.r.t. t partially twice, we get,

$$\frac{\partial K}{\partial t} = 2\beta Cq^2 t + 2A\beta qt \tag{6.43}$$

$$\frac{\partial^2 K}{\partial t^2} = 2\beta Cq^2 + 2A\beta q \tag{6.44}$$

also $$\frac{\partial^2 K}{\partial q\, \partial t} = 4\beta Cqt + 2A\beta t \tag{6.45}$$

Now for total cost to be minimum, the necessary condition is,

$$\left(\beta Ct^2 + \frac{A\alpha}{q^3}\right)(Cq + A)q - \beta t^2 (2Cq + A)^2 > 0$$

or $$\frac{AC\alpha}{q} + \frac{A^2\alpha}{q^2} - 3\beta C^2 q^2 t^2 - A\beta Ct^2 q - \beta A^2 t^2 > 0$$

or $$3\beta C^2 t^2 q^4 + A\beta Ct^2 q^3 + A^2 \beta t^2 q^2 - AC\alpha q - A^2\alpha < 0 \tag{6.46}$$

Also $K(q, t)$ will be minimum if $\frac{\partial^2 K}{\partial t^2} > 0$ at values q^* and t^*, obtained from equation (6.46).

CONCLUDING REMARKS

In the EOQ model considered in this chapter, demand rate depends on stock-level and time both – a more realistic situation. In classical EOQ model in fixed demand with no shortages allowed, results do not depend on unit cost and selling price of the item, while it may depend on item for variable demand.

Some more functional forms of demand rate can be chosen suitably in order to extend the above EOQ model.

CHAPTER–7

INVENTORY MODEL WITH EXPONENTIAL DEMAND AND CONSTANT DETERIORATION WITH SHORTAGES

INTRODUCTION

Inventory problems involving time-variable demand patterns have recieved the attention of several researchers in recent years. Among them **Silver and Meal** [90], **Donaldson** [23], **Ritchie** [85] are worth mentioning. **Mitra** [68] presented a simple procedure for adjusting the economic order quantiity model for the cases for increasing or decreasing linear trend in demand. **Dave and Patel** [26] developed an inventory model with time proportional demand. This model was extended by **Sachan** [98] to cover backlogging option. **Datta and Pal** [32] derived an optimal replacement policy by taking demand rate in power pattern form and variable rate of deterioration. **Goswami and Chaudhuri** [48] developed an EOQ model for deteriorating item with shortages and linear trend in demand.

In the first section of present chapter an inventory model is derived in which demand varies exponentially as time and a constant fraction of on hand inventory deteriorates with time. Shortages also run for a fixed time interval. No deterioration case is also discussed and it is shown that when there be no deterioration, inventory model for deteriorating items reduces to the case of non-deteriorating item. The model is also discussed for probabilistic demand in both the cases. In the second section, an EOQ model with exponential demand for deteriorating items is developed to determine reorder points, shortage intervals over a finite time horiizon so as to keep the total cost to a minimum.

The part of this chapter has been recommended for publication in Indian Journal of Mathematics.

DEVELOPMENT OF THE MODEL

SECTION – 1

DETERIORATION CASE FOR DETERMINISTIC DEMAND

EOQ model is derived under the assumption that replenishment size is constant and replenishment rate is infinite during the time length T of each production cycle. Lead time is zero, shortages are allowed and fully backlogged. Demand rate is $\dfrac{de^{t/T}}{(e-1)\,T}$ at any time t and units deteriorate at a constant rate θ (say).

Let Q be the total amount of inventory produced or purchased at the beginning of each period and after satisfying back orders, let us assume the level of inventory as S (>0). Let d be the demand during period T. Inventory level gradually diminishes during the period $(0, t_1)$, $t_1 < T$, due to the demand and deterioration of the items and ultimately falls to zero at time $t = t_1$. Shortages occur then during the period (t_1, T) which are completely backlogged.

If q(t) be the on hand inventory then differential equations governing the production cycle are,

$$\frac{d}{dt}q(t) + \theta\, q(t) = -\frac{d}{(e-1)\,T}\, e^{t/T}, \qquad 0 \le t \le t_1 \tag{7.1}$$

and

$$\frac{d}{dt}\, q(t) = -\frac{d}{(e-1)\,T}\, e^{t/T}, \qquad t_1 < t \le T \tag{7.2}$$

Solution of equation (7.1) follows as,

$$q(t)\, e^{\theta t} = -\frac{d}{(e-1)T} \int e^{t/T} e^{\theta t}\, dt + B, \text{ where } e^{\theta t} \text{ is integrating factor.}$$

$$= -\frac{d}{(e-1)(1+\theta T)}\, e^{(1+\theta T)\, t/T} + B$$

Since at $t = 0$, $q = S$, we get

$$S = -\frac{d}{(e-1)(1+\theta T)} + B$$

or $$B = S + \frac{d}{(e-1)(1+\theta T)}$$

Putting value of B above, we obtain,

$$q(t) = \frac{d}{(e-1)(1+\theta T)} e^{-\theta T} - \frac{d}{(e-1)(1+\theta T)} e^{t/T} + S\, e^{-\theta t} \tag{7.3}$$

Solution of differential equation (7.2) using condition

$q(t_1) = 0$ is given by

$$q(t) = \frac{d}{(e-1)} \left(e^{t_1/T} - e^{t/T}\right) \tag{7.4}$$

Since at $t = t_1$, $q = 0$, equation (3) implies,

$$S = \frac{d}{(e-1)(1+\theta T)} \left[e^{(1+\theta T)t_1/T} - 1\right] \tag{7.5}$$

Hence deteriorated amount

$$D = S - \frac{d}{(e-1)\,T} \int_0^{t_1} e^{t/T}\, dt$$

$$= S - \frac{d}{(e-1)} \left[e^{t_1/T} - 1\right]$$

$$= \frac{d}{(e-1)(1+\theta T)} \left[e^{(\theta + \frac{1}{T})\, t_1} - 1\right] - \frac{d}{(e-1)} \left[e^{t_1/T} - 1\right] \tag{7.6}$$

Total average number of units in inventory during $(0, t_1)$.

$$I_1(t_1) = \frac{1}{T} \int_0^{t_1} \left[\frac{d}{(e-1)(1+\theta T)} e^{-\theta t} - \frac{d}{(e-1)(1+\theta T)} e^{t/T} + S\, e^{-\theta t}\right] dt$$

$$= \frac{1}{T} \Bigg[- \frac{d}{\theta(e-1)(1+\theta T)} e^{-\theta t_1} - \frac{dT}{(e-1)(1+\theta T)} e^{t_1/T} - \frac{S}{\theta} e^{-\theta t_1}$$

$$+ \frac{d}{\theta(e-1)(1+\theta T)} + \frac{dT}{(e-1)(1+\theta T)} + \frac{S}{\theta}\Bigg]$$

Using equation (7.5) and simplifying above reduces to,

$$I_1(t_1) = \frac{d}{(e-1)(1+\theta T)T}\left[\frac{e^{(1+\theta T)\frac{t_1}{T}}}{\theta} - \frac{e^{t_1/T}}{\theta} - Te^{t_1/T} + T\right] \tag{7.7}$$

Also average number of units in shortages during (t_1, T)

$$I_2(t_1) = \frac{1}{T}\int_{t_1}^{T} \frac{d}{(e-1)}\left[e^{t_1/T} - e^{t/T}\right] dt$$

$$= \frac{d}{(e-1)T}\left[2Te^{t_1/T} - t_1 e^{t_1/T} - eT\right] \tag{7.8}$$

Finally average total cost per unit time is given by

$$K(t_1) = \frac{dC_1}{(e-1)T(1+\theta T)}\left[\frac{e^{(1+\theta T)t_1/T}}{\theta} - \frac{e^{t_1/T}}{\theta} - T\,e^{t_1/T} + T\right]$$

$$- \frac{dC_2}{(e-1)T}\left[2Te^{t_1/T} - t_1 e^{t_1/T} - eT\right] + \frac{dC}{(e-1)T(1+\theta T)}\left[e^{(\theta+\frac{1}{T})t_1} - 1\right]$$

$$- \frac{dC}{(e-1)T}\left[e^{t_1/T} - 1\right] \tag{7.9}$$

where C is the cost of each unit and C_1, C_2 are holding and shortage cost per unit per unit time respectively.

Condition for which to be monimum is,

$$\frac{d}{dt_1} K(t_1) = 0$$

This gives,

$$\frac{dC_1}{(e-1)T(1+\theta T)}\left[\frac{(1+\theta T)}{\theta T} e^{(1+\theta T)t_1/T} - e^{t_1/T} - \frac{1}{T} e^{t_1/T}\right]$$

$$- \frac{dC_2}{(e-1)\,T}\left[e^{t_1/T} - \frac{t_1}{T} e^{t_1/T}\right] + \frac{dC}{(e-1)T^2} e^{(1+\theta T)t_1/T} - \frac{dC}{(e-1)T^2} e^{t_1/T} = 0$$

or $$\frac{d}{(e-1)T} e^{t_1/T}\left[\left\{\frac{e^{\theta t_1}}{\theta T}-\frac{1}{(1+\theta T)}-\frac{1}{\theta T}(1+\theta T)\right\}C_1-\left(1-\frac{t_1}{T}\right)C_2+\frac{1}{T}(e^{\theta t_1}-1)C\right]=0 \tag{7.10}$$

which is a transcedental equation in t_1 and can be solved using any iterative method to get $t_1 = t_1^*$ as a solution. The total cost will be minimum at $t_1 = t_1^*$ if $\frac{d^2K}{dt_1^2} > 0$ at $t_1 = t_1^*$.

DETERIORATION CASE FOR PROBABILISTIC DEMAND

In this case, it is assumed that the demand during the period (0, T) is a random variable with probability density function $f(x)$ $(0 < x < \infty)$ and demand rate is $\frac{x}{(e-1)} e^{t/T}$.

Case I : When no shortages occur.

Let $q_{1x}(t)$ be inventory level of the system at any time t $(0 \le t \le T)$, then differential equation which would describe the system is,

$$\frac{d}{dt} q_{1x}(t) + \theta q_{1x}(t) = -\frac{x}{(e-1)T} e^{t/T}, \qquad 0 \le t \le T \tag{7.11}$$

and its solution is,

$$q_{1x}(t) = \frac{x}{(e-1)(1+\theta T)} e^{-\theta t} - \frac{x}{(e-1)(1+\theta T)} e^{t/T} + S\, e^{-\theta t}, \quad 0 \le t \le T \tag{7.12}$$

where $S\,(>0)$ is the expected stock on hand after meeting back orders.

Since there is no shortages, we have,

$$q_{1x}(T) \ge 0$$

or $$\frac{x}{(e-1)(1+\theta T)} e^{-\theta T} - \frac{xe}{(e-1)(1+\theta T)} + Se^{-\theta T} \ge 0,$$ using equation (7.12).

or $$\frac{x}{(e-1)(1+\theta T)} (e - e^{-\theta T}) \le Se^{-\theta T}$$

$$\text{or} \quad x \le \frac{S(e-1)(1+\theta T)\,e^{-\theta T}}{(e-e^{-\theta T})}$$

$$\text{or} \quad x \le \frac{S(e-1)(1+\theta T)}{(e^{1+\theta T}-1)} = S_1 \tag{7.13}$$

The average number of items carried in inventory per unit time,

$$H_1(x) = \frac{1}{T}\int_0^T q_{1x}(t)\,dt, \qquad x \le S_1$$

$$= \frac{x}{(e-1)T(1+\theta T)}\left[\frac{e^{(\theta T+1)}}{\theta} - \frac{e}{\theta} - eT + T\right]$$

$$= \frac{x}{(e-1)(1+\theta T)} \tag{7.14}$$

Average number of items deteriorated per unit time,

$$D_1(x) = \frac{1}{T}\left[S - x - q_{1x}(T)\right]$$

$$= \frac{1}{T}\left[S - x - \frac{x}{(e-1)(1+\theta T)}e^{-\theta T} + \frac{x}{(e-1)(1+\theta T)}e - Se^{-\theta T}\right]$$

$$= \frac{1}{T}\left[S(1-e^{-\theta T}) + \frac{x}{(e-1)(1+\theta T)}(e - e^{\theta T}) - x\right]$$

$$= S\theta + \frac{x}{(e-1)(1+\theta T)}(e - e^{-\theta T}) - \frac{x}{T}; \qquad x \le S_1 \tag{7.15}$$

Average shortage

$$G_1(x) = 0 \tag{7.16}$$

Case II : When shortages occur.

Let in this case system carries inventory during the period $(0, t_1)$ and shortages occur for the remaining period (t_1, T) of the cycle. If $q_{2x}(t)$ be inventory level at any time t, then differential equations governing the system would be

$$\frac{d}{dt}q_{2x}(t) + \theta q_{2x}(t) = -\frac{x}{(e-1)T}e^{t/T}, \qquad 0 \le t \le t_1 \tag{7.17}$$

and $$\frac{d}{dt} q_{2x}(t) = -\frac{x}{(e-1)T} e^{t/T}, \qquad t_1 < t \leq T \tag{7.18}$$

Solutions to which are,

$$q_{2x}(t) = \frac{x}{(e-1)(1+\theta T)} e^{-\theta T} - \frac{x}{(e-1)(1+\theta T)} e^{t/T} + Se^{-\theta t}, \quad 0 \leq t \leq t_1 \tag{7.19}$$

and $$q_{2x}(t) = \frac{x}{(e-1)} (e^{t_1/T} - e^{t/T}), \qquad t_1 < t \leq T \tag{7.20}$$

Since shortages occur, we must have,

$q_{2x}(T) < 0$

or $x > S_1$, where S_1 is given by equation (7.13).

Again, $q_{2x}(t_1) = 0$, this gives,

$$\frac{xe^{-\theta t_1}}{(e-1)(1+\theta T)} - \frac{x\, e^{t_1/T}}{(e-1)(1+\theta T)} + S\, e^{-\theta t_1} = 0$$

or $$\frac{x}{(e-1)(1+\theta T)} e^{-\theta t_1} \left[e^{(1+\theta T)t_1/T} - 1 \right] = S\, e^{-\theta t_1}$$

or $$t_1 = \frac{T}{(1+\theta T)} \log \left[1 + \frac{S\,(e-1)(1+\theta T)}{x} \right] \tag{7.21}$$

The average number of items carried in inventory per unit time,

$$H_2(x) = \frac{1}{T} \int_0^{t_1} q_{2x}(t)\, dt, \qquad x > S_1$$

$$= \frac{x}{(e-1)(1+\theta T)\, T} \left[\frac{e^{t_1/T}}{\theta} (e^{\theta t_1} - 1) - T(e^{t_1/T} - 1) \right] \tag{7.22}$$

Average amount of inventory that deteriorate per unit time,

$$D_2(x) = \frac{1}{T} \left[S - \int_0^{t_1} \frac{x}{(e-1)T} e^{t/T}\, dt \right], \qquad x > S_1$$

$$= \frac{1}{T} \left[S - \frac{x}{(e-1)} (e^{t_1/T} - 1) \right] \tag{7.23}$$

and average storages per unit time,

$$G_2(x) = \frac{1}{T} \int_{t_1}^{T} \frac{x}{(e-1)} \left[e^{t_1/T} - e^{t/T}\right] dt , \qquad x > S_1$$

$$= \frac{x}{(e-1)\,T} \left[2Te^{t_1/T} - t_1 e^{t_1/T} - eT\right] \tag{7.24}$$

Finally expected total cost of the system per unit time becomes,

$$K(t_1, S) = C \int_0^{S_1} D_1(x)\, f(x)\, dx + C \int_{S_1}^{\infty} D_2(x)\, f(x)\, dx + C_1 \int_0^{S_1} H_1(x)\, f(x)\, dx$$

$$+ C_1 \int_{S_1}^{\infty} H_2(x)\, f(x)\, dx - C_2 \int_0^{S_1} G_1(x)\, f(x)\, dx - C_2 \int_{S_1}^{\infty} G_2(x)\, f(x)\, dx \tag{7.25}$$

Now substituting the value of $D_1(x)$, $D_2(x)$, $H_1(x)$, $H_2(x)$, $G_1(x)$ and $G_2(x)$ from equations (7.15), (7.23), (7.14), (7.22), (7.16) and (7.24) respectively in cost equation (7.25), we have,

$$K(t_1, S) = \frac{C}{T} \int_0^{(e^{1+\theta T} - 1)} \left[S\theta + \frac{x(e - e^{-\theta T})}{(e-1)(1+\theta T)\,T} - \frac{x}{T}\right] f(x)\, dx +$$

$$\frac{C}{T} \int_{\frac{S(e-1)(1+\theta T)}{(e^{1+\theta T}-1)}}^{\infty} \left[S - \frac{x}{(e-1)} (e^{t_1/T} - 1)\right] f(x)\, dx$$

$$+ \frac{C_1}{(e-1)(1+\theta T)} \int_0^{\frac{S(e-1)(1+\theta T)}{(e^{1+\theta T}-1)}} x\, f(x)\, dx$$

$$+ \frac{C_1}{(e-1)(1+\theta T)\,T} \int_{\frac{S(e-1)(1+\theta T)}{(e^{\theta T}-1)}}^{\infty} x \left[\frac{e^{t_1/T}}{\theta} (e^{\theta t_1} - 1) - T(e^{t_1/T} - 1)\right] f(x)\, dx$$

$$- \frac{C_2}{(e-1)\,T} \int_{S\frac{(e-1)(1+\theta T)}{(e^{\theta T}-1)}}^{\infty} x \left[2 + e^{t_1/T} - t_1 e^{t_1/T} - eT\right] f(x)\, dx \tag{7.26}$$

If the probabilty density function f(x) is known then from equation (7.21) and above equation (7.26), the expected total cost per unit time K(S) of the system can be evaluated. The necessary condition for which to be minimum is

$$\frac{d}{dS} K(S) = 0 \tag{7.27}$$

The solution $S = S^*$ of equation (7.27) will be optimum value of S if $\frac{d^2}{dS^2} K(S) > 0$ at $S = S^*$.

NON DETERIORATION CASE FOR DETERMINISTIC DEMAND

In deriving EOQ model here, all the assumptions and conditions are same as in deterministic model of this chapter except that deterioration does not take place.

Therefore, differential equations describing the system are

$$\frac{d}{dt} q(t) = -\frac{d}{(e-1)T} e^{t/T}, \qquad o \le t \le t_1 \tag{7.28}$$

and

$$\frac{d}{dt} q(t) = -\frac{d}{(e-1)T} e^{t/T}, \qquad t_1 \le t \le T \tag{7.29}$$

Solutions of which follow as,

$$q(t) = \frac{d}{(e-1)} (1 - e^{t/T}) + S \tag{7.30}$$

and

$$q(t) = \frac{d}{(e-1)} (e^{t_1/T} - e^{t/T}) \tag{7.31}$$

Since $q(t_1) = 0$, this gives from equation (7.30),

$$S = \frac{d}{(e-1)} (e^{t_1/T} - 1) \tag{7.32}$$

Total number of units in inventory

$$= \int_0^{t_1} q(t)\, dt$$

$$= \frac{d}{(e-1)}\left[t_1 e^{t_1/T} - T e^{t_1/T} + T\right], \text{ using equations (7.30) and (7.32)}$$

(7.33)

and total number of units running in shortages,

$$= \int_{t_1}^{T} q(t)\,dt = \frac{d}{(e-1)}\left[2T e^{t_1/T} - t_1 e^{t_1/T} - eT\right] \tag{7.34}$$

Hence total average cost of the system per unit time is given by,

$$K(t_1) = \frac{C_1}{T}\int_0^{t_1} q(t)\,dt - \frac{C_2}{T}\int_{t_1}^{T} q(t)\,dt$$

$$= \frac{dC_1}{(e-1)T}\left(t_1 e^{t_1/T} - T e^{t_1/T} + T\right) - \frac{dC_2}{(e-1)T}\left(2T e^{t_1/T} - t_1 e^{t_1/T} - eT\right)$$

(7.35)

For this total cost $K(t_1)$ to be minimum, necessary condition is,

$\frac{d}{dt_1} K(t_1) = 0$ which implies,

$$\frac{d}{(e-1)T^2}\left[(C_1 + C_2)t_1 - C_2 T\right] e^{t_1/T} = 0$$

or $$t_1 = \left(\frac{C_2}{C_1 + C_2}\right) T \tag{7.36}$$

This value t_1 is optimal value of time t as

$$\frac{d^2}{dt_1^2} K(t_1) > 0 \text{ at } t_1 = \left(\frac{C_2}{C_1 + C_2}\right) T.$$

Therefore, $t_1^* = \left(\frac{C_2}{C_1 + C_2}\right)T$ and other optimal quantities are,

$$Q^* = \frac{d}{(e-1)}\left(e^{t_1^*/T} - 1\right) \text{ and } Q^* = d$$

Again equation (7.10) can be rewritten as,

$$\frac{1}{\theta T\left(\theta + \frac{1}{T}\right)}\left\{(1 + \theta T)\, e^{\theta t_1} - \theta T - 1\right\}C_1 - \left(T - \frac{t_1}{T}\right)C_2 + \frac{1}{T}\,(e^{\theta t_1} - 1)C = 0$$

or $$\frac{1}{\theta T\,(1 + \theta T)}\left\{(\theta T + 1)(1 + \theta t_1) - \theta T - 1\right\}C_1 - \left(T - \frac{t_1}{T}\right)C_2 + \frac{1}{T}\,(e^{\theta t_1} - 1)\,C = 0$$

neglecting higher order terms of θ as $0 < \theta < 1$.

In particular if $\theta = 0$, above expression reduces to,

$$\frac{C_1}{T}\,t_1 - \left(1 - \frac{t_1}{T}\right)C_2 = 0$$

or $t_1 = \left(\frac{C_2}{C_1 + C_2}\right)T$, which is same as given by equation (7.36)

This shows that when there be no deterioration, the inventory model for deteriorating items reduces to that for non-deteriorating items.

NON DETERIORIORATION FOR PROBABILISTIC DEMAND

Here EOQ model is derived with same assumptions and condition as in case of probabilistic model for deteriorating items of this chapter except that the deterioration does not occur.

In no shortage case differential equation describing the system is,

$$\frac{d}{dt}\,q_1(x) = -\,\frac{x}{(e - 1)\,T}\,e^{t/T}, \qquad 0 \le t \le T \tag{7.37}$$

and its solution becomes,

$$q_{1x}(t) = \frac{x}{(1 - e)}\,(1 - e^{t/T}) + S \tag{7.38}$$

Also in this case,

$q_{1x}(T) \ge 0$, which gives,

$$x \le S \tag{7.39}$$

where S is given by equation (7.32).

Average number of items in inventory per unit time

$$H_1(x) = \frac{1}{T}\int_0^T q_{1x}(t)\,dt$$

$$= \frac{x}{(e-1)} \tag{7.40}$$

and average shortage per unit time

$$G_1(x) = 0$$

Now when shortages occur, differential equations for the system are similar to equations (7.28) and (7.29) and their solutions are

$$q_{2x}(t) = \frac{x}{(e-1)}(1 - e^{t/T}) + S, \qquad 0 \le t \le t_1 \tag{7.41}$$

and $$q_{2x}(t) = \frac{x}{(e-1)}(e^{t_1/T} - e^{t/T}), \qquad t_1 < t \le T \tag{7.42}$$

Since shortages occur

$q_{2x}(T) < 0$ which gives,

$x > S,$

Also $q_{2x}(t_1) = 0$ from equation (7.42), this gives,

$$\frac{x}{(e-1)}(e^{t_1/T} - 1) = S$$

or $$e^{t_1/T} = 1 + \frac{S(e-1)}{x}$$

or $$t_1 = T\log\left[1 + \frac{S(e-1)}{x}\right] \tag{7.43}$$

The average number of items in the system per unit time

$$H_2(x) = \frac{1}{T}\int_0^{t_1} q_{2x}(t)\,dt$$

$$= \frac{x}{(e-1)T}\left[t_1 e^{t_1/T} - T e^{t_1/T} + T\right], \qquad x > S \tag{7.44}$$

And average shortage per unit time

$$G_2(x) = \frac{1}{T}\int_{t_1}^{T} q_{2x}(t)\,dt$$

$$= \frac{x}{(e-1)T}\left[2T e^{t_1/T} - t_1 e^{t_1/T} - eT\right] \tag{7.45}$$

Therefore expected total cost of the system per unit time

$$K(t_1, S) = \frac{C_1 T}{e}\int_0^{Se/T(e-1)} x\,f(x)\,dx + \frac{C_1}{e}\int_{Se/T(e-1)}^{\infty} x\left[t_1 e^{t_1/T} - T e^{t_1/T} + T\right] f(x)\,dx$$

$$- \frac{C_2 T}{e}\int_{Se/T(e-1)}^{\infty} x\left[2t\, e^{t_1/T} - t_1 e^{t_1/T} - eT\right] f(x)\,dx \tag{7.46}$$

If the probability density function f(x) is known explicitly, then using equation (7.43) in equation (7.46) total expected cost K(S) can be estimated. For which to be minimum, condition is

$$\frac{d}{dS}K(S) = 0 \tag{7.47}$$

The solution $S = S^*$ of equation (7.47) will be optimal if

$$\frac{d^2}{dS^2}K(S) > 0 \text{ at } S = S^*.$$

SECTION – 2

AN INVENTORY REPLENISHMENT POLICY

(Different Approah)

In this section an inventory replenishment policy over a fixed planning period H for a deteriotating item with constant rate θ with an exponential trend in demand, i.e. $r(q) = \alpha e^{\beta t}$, α, β are constants with shortages is considered.

The total time horizon H has been devided into n equal parts of length T so that $T = \frac{H}{n}$. The recorder times over the time horizon H will be jT $(j = 0, 1, 2, \ldots, n-1)$. Initial and final inventories are both zero. Let the period for which there is no shortage in each interval $[jT, (j+1)T]$ is a fraction of scheduling period T and is equal to RT $(0 < R < 1)$. Shortages occur at times $(R + j)T$ $(j = 0, 1, \ldots, n-2)$ where $jT < (R+j)T < (j+1)T$, $j = 0, 1, 2, \ldots, n-2$. Last replenishment occurs at time $(n-1)T$ and shortages are not allowed in the last period $[(n-1)T, H]$. Our purpose is to derive the optimal reorder and shortage points and hence to determine the optimal values of n and R, which minimize the total cost over the time horizon [0, H]. The order quantity at each order point is the amount needed to satisfy the demand required for the relevant period excluding the shortage period and the amount required to account for the deterioration in the no shortage part of the said period. Thus if Q_j units are ordered at time $(j-1)T$, then

$$Q_j = \alpha \int_{(j-1)T}^{(R+j-1)T} e^{(\theta+\beta)t}\, dt$$

$$= \frac{\alpha}{(\theta+\beta)} \left[e^{(\theta+\beta)t} \right]_{(j-1)T}^{(R+j-1)T}$$

$$= \frac{\alpha}{(\theta+\beta)} \left[e^{(\theta+\beta)(R+j-1)T} - e^{(\theta+\beta)(j-1)T} \right]$$

$$= \frac{\alpha}{(\alpha+\beta)} e^{(\alpha+\beta)(j-1)T} \left[e^{(\theta+\beta)RT} - 1 \right] \tag{7.48}$$

for $j = 1, 2, \ldots, (n-1)$

Since shortages are not allowed in the last period, so Q_j for $j = n$ should be calculated seperately, and

$$Q_n = \alpha \int_{(n-1)T}^{H} e^{(\theta+\beta)t}\,dt$$

$$= \frac{\alpha}{(\theta+\beta)}\left[e^{(\theta+\beta)H} - e^{(\theta+\beta)(n-1)T}\right] \qquad (4.49)$$

Let $\overline{Q}_j$ be the number of units required in [jT, (j + 1)T] (j=1, 2, . . . , (n − 1)), when there is no deterioration, then

$$\overline{Q}_j = \alpha \int_{(j-1)T}^{(R+j-1)T} e^{\beta t}\,dt$$

$$= \frac{\alpha}{\beta}\left[e^{\beta(R+j-1)T} - e^{\beta(j-1)T}\right]$$

$$= \frac{\alpha}{\beta} e^{\beta(j-1)T}\left[e^{\beta RT} - 1\right] \qquad j = 1,\ 2,\ .\ .\ .\ .\ ,\ (n-1) \qquad (7.50)$$

and $\overline{Q}_n = \alpha \int_{(n-1)T}^{H} e^{\beta t}\,dt$

$$= \frac{\alpha}{\beta}\left[e^{\beta H} - e^{\beta(n-1)T}\right] \qquad (7.51)$$

Hence number of deteriorated items in the cycle [jT, (R + j) T] j=0, 1, ... , (n−1), can be given by

$$D_j = Q_j - \overline{Q}_j$$

$$= \frac{\alpha}{(\alpha+\beta)} e^{(\alpha+\beta)(j-1)T}\left[e^{(\theta+\beta)T} - 1\right] - \frac{\alpha}{\beta} e^{(j-1)T}\left[e^{\beta RT} - 1\right]$$

$$= \alpha\, e^{\beta(j-1)T}\left[\frac{e^{\theta(j-1)T}}{(\theta+\beta)}\left\{e^{(\theta+\beta)RT} - 1\right\} - \frac{1}{\beta}\left\{e^{\beta RT} - 1\right\}\right] \qquad (7.52)$$

for j=1, 2, ... , (n − 1)

and similarly

$$D_n = Q_n - \overline{Q}_n$$

$$= \frac{\alpha}{(\theta+\beta)}\left[e^{(\theta+\beta)H} - e^{(\theta+\beta)(n-1)T}\right] - \frac{\alpha}{\beta}\left[e^{\beta H} - e^{\beta(n-1)T}\right]$$

$$= \alpha \left[e^{\beta H} \left\{ \frac{e^{\theta H}}{(\theta + H)} - \frac{1}{\beta} \right\} - e^{\beta(n-1)T} \left\{ \frac{e^{\theta(n-1)T}}{(\theta + \beta)} - \frac{1}{\beta} \right\} \right] \tag{7.53}$$

The inventory holding cost over the period $[jT, (j+1)T]$ $j = 0, 1, \ldots, (n-1)$, is given by $C_1 R_j$, where R_j is

$$R_j = \alpha \int_{(j-1)T}^{(R+j-1)T} [t - (j-1)T]\, e^{(\theta + \beta)t}\, dt$$

$$= \alpha \left[\left\{ t - \frac{(j-1)T}{(\theta + \beta)} \right\} e^{(\theta+\beta)t} - \frac{e^{(\theta+\beta)t}}{(\theta+\beta)^2} \right]_{(j-1)T}^{(R+j-1)T}$$

$$= \alpha \left[\left\{ \frac{RT}{(\theta+\beta)} - \frac{1}{(\theta+\beta)^2} \right\} e^{(\theta+\beta)(R+j-1)T} + \frac{1}{(\theta+\beta)^2} e^{(\theta+\beta)(j-1)T} \right]$$

$$R_j = \frac{\alpha\, e^{(\theta+\beta)(j-1)T}}{(\theta+\beta)^2} \left[\{(\theta+\beta) RT - 1] \; e^{(\theta+\beta)RT} + 1 \right] \tag{7.54}$$

$$j = 1, 2, \ldots, (n-1)$$

and $$R_n = \alpha \int_{(n-1)T}^{H} \{t - (n-1)T\}\, e^{(\theta+\beta)t}\, dt$$

$$= \alpha \left[\left\{ \frac{t - (n-1)T}{(\theta+\beta)} \right\} e^{(\theta+\beta)t} - \frac{e^{(\theta+\beta)t}}{(\theta+\beta)^2} \right]_{(n-1)T}^{H}$$

$$= \alpha \left[\left\{ \frac{H - (n-1)T}{(\theta+\beta)} - \frac{1}{(\theta+\beta)^2} \right\} e^{(\theta+\beta)H} + \frac{e^{(\theta+\beta)(n-1)T}}{(\theta+\beta)^2} \right]$$

$$= \frac{\alpha}{(\theta+\beta)^2} \left[\{(H - (n-1)T(\theta+\beta) - 1\} e^{(\theta+\beta)H} + e^{(\theta+\beta)(n-1)T} \right] \tag{7.55}$$

Since there are n production runs, total set up cost is nA.

If S_j be the numbers of units in shortages, in $[(R+j-1)T, jT]$, then

$$S_j = \alpha \int_{(R+j-1)T}^{jT} (jT - t)\, e^{\beta t}\, dt$$

$$= \alpha \left[\frac{(jT - t)\, e^{\beta t}}{\beta} + \frac{1}{\beta^2} e^{\beta t} \right]_{(R+j-1)T}^{jT}$$

$$= \frac{\alpha}{\beta^2} \left[e^{\beta jT} + \{\beta(R-1)T - 1\} \right] e^{\beta(R+j-1)T}$$

$$= \frac{\alpha}{\beta^2} e^{\beta jT} \left[\{\beta(R-1)T - 1\}\, e^{\beta(R-1)T} + 1 \right] \tag{7.56}$$

$j = 1, 2, \ldots, (n-1)$

Therefore the total cost K over the horizon H is given by

$$K = nA + C_1 \sum_{j=1}^{n-1} R_j + C \sum_{j=1}^{n-1} D_j + C_1 R_n + CD_n + C_2 \sum_{j=1}^{n-1} S_j \tag{7.57}$$

$$= nA + \frac{C_1 \alpha}{(\theta+\beta)^2} \sum_{j=1}^{n-1} e^{(\theta+\beta)(j-1)T} \left[\{(\theta+\beta)RT - 1\}\, e^{(\theta+\beta)RT} + 1 \right]$$

$$+ C\alpha \sum_{j=1}^{n-1} e^{\beta(j-1)T} \left[\frac{e^{\theta(j-1)T}}{(\theta+\beta)} \left\{ e^{(\theta+\beta)RT} - 1 \right\} - \frac{1}{\beta} \left\{ e^{\beta RT} - 1 \right\} \right]$$

$$+ \frac{C_1 \alpha}{(\theta+\beta)^2} \left[\{(H - \overline{n-1}\, T)(\theta+\beta) - 1\}\, e^{(\theta+\beta)H} + e^{(\theta+\beta)(n-1)T} \right]$$

$$+ C\alpha \left[e^{\beta H} \left\{ \frac{e^{\theta H}}{(\theta+\beta} - \frac{1}{\beta} \right\} - e^{\beta(n-1)T} \left\{ \frac{e^{\theta(n-1)T}}{(\theta+\beta)} - \frac{1}{\beta} \right\} \right]$$

$$+ \frac{C_2 \alpha}{\beta^2} \sum_{j=1}^{n-1} e^{\beta jT} \left[\{\beta(R-1)T - 1\}\, e^{\beta(R-1)T} + 1 \right] \tag{7.58}$$

It is obvious that above costs function K (n, R) is bi-variate in R and n, one continuous and other discrete respectively.

For a given value of n, the necessary condition for K to be minimum

DISCUSSION

An EOQ model is suggested here in which demand depends on time in exponential manner and units in inventory deteriorate at a constant rate. Shortages are existing for a prescribed time period. Deterministic and probabilistic, both the models are established. The model is also worked out for non-deteriorating items in both the cases of demand. Total minimum cost and expected total cost per unit time for the system have been derived. It is shown that if deterioration does not affect the items in the inventory, then the model developed in the first case reduce to the model for non-deteriorating items derived later.

In second section an EOQ model is derived with different exponential demand rate for deteriorating items with shortages to determine reorder points, interval between two successive reorders and shortage intervals over a finite time horizon.

These inventory models can be further extended for variable deterioration and (or) finite rate of replenishment.

CHAPTER – 8

AN EOQ MODEL WITH EXPONENTIAL DEMAND AND SHORTAGES FOR VARIABLE DETERIORATING ITEMS

INTRODUCTION

In formulating inventory models, two facets of the problem have been of growing interest. One being the deterioration of items and the other variation in the demand rate. Among researchers considering inventory models in deteriorating items, **Shah and Jaiswal** [91] considered the rate of deterioration to be uniform, **Covert and Philip** [16] produced an EOQ model for items with variable rate of deterioration. **Misra** [67] used a two parameter Weibull distribution to fit the deterioration rate.

The present chapter deals with the generalisation of EOQ model with exponential demand for constantly deteriorating items derived in previous chapter. In the EOQ model presented here, demand rate is taken in exponential form and a special form of Weibull density function is chosen in order to make the problem mathematically tractable. Deterministic as well as probabilistic cases of demands are considered allowing shortages.

DERIVATION OF THE MODEL

DETERMINISTIC DEMAND CASE

It is assumed that replenishment size is constant and production is instantaneous during the prescribed time period T of each cycle. Lead time is zero, shortages are permitted and completely accumulated. Demand rate is $\dfrac{d}{(e-1)\,T}\, e^{t/T}$ at any time t. A variable fraction $\theta(t)$ of the on hand inventory deteriorates per unit time and is of the form $\theta = \theta_0\, t$, where θ_0 is a constant with the condition $0 < \theta_0 < 1$, $t > 0$.

Let Q be the quantity produced or purchased at the beginning of each production cycle and after satisfying back orders, let an amount S remains as a initial inventory. Let d be the demand during the time period T. Inventory level gradually reduces and finally becomes zero at $t = t_1 < T$, then shortages occur during (t_1, T) and are fully backlogged.

If q(t) be the current stock level at any time t, their differential equations which the on hand inventory q(t) must satisfy in two different parts of the cycle time T are as following :

$$\frac{d}{dt} q(t) + \theta_0 t\, q(t) = -\frac{d}{(e-1)T} e^{t/T}, \qquad 0 \le t \le t_1 \tag{8.1}$$

$$\frac{d}{dt} q(t) = -\frac{d}{(e-1)T} e^{t/T}, \qquad t_1 < t \le T \tag{8.2}$$

Solution to differntial equation (8.1) follows as

$$q(t)\, e^{\theta_0 t^2/2} = -\frac{d}{(e-1)T} \int e^{\theta_0 t^2/2} e^{t/T} dt + B,$$

where $e^{\theta_0 t^2/2}$ is integrating factor and B is constant of integration.

$$= -\frac{d}{(e-1)T} \int \left(e^{t/T} + \frac{\theta_0}{2} t^2 e^{t/T}\right) dt + B$$

neglecting higher order terms of θ as $0 < \theta < 1$

$$= -\frac{d}{(e-1)T} \left[t\, e^{t/T} + \frac{\theta_0}{2}\left\{T\, t^2 e^{t/T} - 2T \int t\, e^{t/T} dt\right\}\right] + B$$

$$= -\frac{d}{(e-1)T} \left[T\, e^{t/T} + \frac{\theta_0}{2} T\, t^2 e^{t/T} - \theta_0 T^2 t\, e^{t/T} + \theta_0 T^3 e^{t/T}\right] + B$$

$$= -\frac{d}{(e-1)} \left[\frac{\theta_0}{2} t^2 - \theta_0 T t + 1 + \theta_0 T^2\right] e^{t/T} + B$$

Since at $t = 0$, $q = S$, we get

$$B = S + \frac{d}{(e-1)} (1 + \theta_0 T^2)$$

Putting value of B above, solution of differential equation (8.1) becomes,

$$= \int_0^{t_1} q(t)\,dt$$

$$= \frac{d}{(e-1)}\left[e^{t/T}\left(-\frac{\theta_0 t^3}{6} + \frac{\theta_0 t_1^2 t}{2} - \theta_0 T t_1 t + t + \theta_0 T^2 t\right)\Bigg|_0^{t_1}\right.$$

$$\left. - \left\{(-\theta_0 T t + 1 + \theta_0 T^2)\, T e^{t/T} + \theta_0 T^3 e^{t/T}\right\}\Bigg|_0^{t_1}\right]$$

$$= \frac{d}{(e-1)}\left[e^{t_1/T}\left(-\frac{\theta_0 t_1^3}{6} + \frac{\theta_0 t_1^3}{2} - \theta_0 T t_1^2 + t_1 + \theta_0 T^2 t_1\right)\right.$$

$$\left. - \left\{\left(-\theta_0 T t_1 + 1 + \theta_0 T^2\right) T e^{t_1/T} + \theta_0 T^3 e^{t_1/T} - (1+\theta_0 T^2)\,T - \theta_0 T^3\right\}\right]$$

$$= \frac{d}{(e-1)}\left[e^{t_1/T}\left\{\frac{\theta_0 t_1^3}{3} - \theta_0 T t_1^2 + (1+2\theta_0 T^2)\, t_1 - T(1+2\theta_0 T^2)\right\} + T(1+2\theta_0 T^2)\right] \tag{8.8}$$

Number of units running in shortages

$$= \int_{t_1}^{T} q(t)\,dt$$

$$= \frac{d}{(e-1)}\left[t\, e^{t/T} - T e^{t/T}\right]_{t_1}^{T}$$

$$= \frac{d}{(e-1)}\left(2T e^{t_1/T} - t_1 e^{t_1/T} - eT\right) \tag{8.9}$$

Finally average total cost per unit time is given by

$$K(t_1) = \frac{CD}{T} + \frac{C_1}{T}\int_0^{t_1} q(t)\,dt - \frac{C_2}{T}\int_{t_1}^{T} q(t)\,dt \tag{8.10}$$

where C is the cost of each item, C_1 and C_2 are holding and shortage costs per unit per unit time respectively.

Substituting the values from equations (8.6), (8.8) and (8.9) cost equation (8.10) reduces to,

$$K(t_1) = \frac{d\,C}{(e-1)\,T}\left[e^{t_1/T}\theta_0\left(t_1^2 - T\,t_1 + T^2\right) - \theta_0 T^2\right]$$

$$+ \frac{d\,C_1}{(e-1)\,T}\left[\left\{e^{t_1/T}\left(\frac{\theta_0 t_1^3}{3} - \theta_0 T\,t_1^2 + (1+2\theta_0 T^2)\,t_1 - T(1+2\theta_0 T^2)\right)\right\} + T(1+2\theta_0 T^2)\right]$$

$$+ \frac{d\,C_2}{(e-1)\,T}\left(2Te^{t_1/T} - t_1 e^{t_1/T} - e\,T\right) \qquad (8.11)$$

Now the total cost will be minimum if

$\frac{d}{dt_1}K(t_1) = 0$, which gives,

$$\frac{d\,C}{(e-1)\,T}\left[\frac{\theta_0 e^{t_1/T}}{T}\left(t_1^2 - T\,t_1 + T^2\right) + \theta_0 e^{t_1/T}(t_1 - T)\right]$$

$$+ \frac{d\,C_1}{e}\left[\left\{\frac{e^{t_1/T}}{T}\left(\frac{\theta_0 t_1^3}{3} - \theta_0 T\,t_1^2 + (1+2\theta_0 T^2)\,t_1 - T(1+2\theta_0 T^2)\right)\right\}\right.$$

$$\left. + e^{t_1/T}\left\{\theta_0 t_1^2 - 2\theta_0 T\,t_1 + 1 + 2\theta_0 T^2\right\}\right] - \frac{d\,C_2}{e}\left[2e^{t_1/T} - e^{t_1/T} - \frac{t_1}{T}e^{t_1/T}\right] = 0$$

or $$\frac{d\,C}{(e-1)\,T^2}\frac{\theta_0}{2}t_1^2 e^{t_1/T} + \frac{d\,C_1}{(e-1)\,T^2}\left(\frac{\theta_0 t_1^3}{3} + t_1\right)e^{t_1/T} - \frac{d\,C_2}{(e-1)\,T^2}(T - t_1)\,e^{t_1/T} = 0$$

or $$\frac{d}{(e-1)\,T^2}e^{t_1/T}\left[\frac{\theta_0}{3}C_1 t_1^3 + \frac{\theta_0}{2}C t_1^2 + (C_1 + C_2)\,t_1 - T\,C_2\right] = 0$$

or $$\frac{\theta_0}{3}C_1 t_1^3 + \frac{\theta_0}{2}C\,t_1^2 + (C_1 + C_2)\,t_1 - T\,C_2 = 0 \qquad (8.12)$$

Equation (8.12) is a cubic equation in t_1 and can be solved by Cardon's method to get one positive real root $t_1 = t_1^*$. The total cost $K(t_1)$ will be minimum provided

$\frac{d^2 K(t_1)}{dt_1^2} > 0$ at $t_1 = t_1^*$, which is verified.

Therefore minimum average cost is $K(t_1^*)$ and other optimum quantities are

$$S^* = \frac{d}{(e-1)}\left[e^{t_1^*/T}\left(\frac{\theta_0 t_1^{*2}}{2} - \theta_0 T t_1^* + 1 + \theta_0 T^2\right) - (1 + \theta_0 T^2)\right], \tag{8.13}$$

and $$Q^* = d + \frac{d}{(e-1)}\theta_0\left[e^{t_1^*/T}\left(\frac{t_1^2}{2} - T t_1^* + T^2\right) - T^2\right] \tag{8.14}$$

In particular if $\theta = 0$, i.e. there be no deterioration the optimal condition (8.12) implies,

$$t_1^* = \left(\frac{C_1}{C_1 + C_2}\right) T \tag{8.15}$$

and other optimal quantities reduce to,

$$S^* = \frac{d}{(e-1)}\left(e^{t_1^*/T} - 1\right) \tag{8.16}$$

and $$Q^* = d \tag{8.17}$$

Expressions (8.15), (8.16) and (8.17) are the same given by EOQ models for non-deteriorating items, derived in seventh chapter.

PROBABILISTIC DEMAND CASE

Let demand during the period (0, T) be a random variable x with probability density function f(x) ($0 < x < d$) and demand follows exponential pattern with demand rate $\frac{x}{(e-1)T}e^{t/T}$. Two cases exist

Case I : When shortage does not occur.

Let $q_{1x}(t)$ be inventory level of the system at any time t, then system can be described mathematically as,

$$\frac{d}{dt}q_{1x}(t) + \theta_0 t\, q_{1x}(t) = -\frac{x}{(e-1)T}e^{t/T}; \qquad 0 \le t \le T \tag{8.18}$$

With solution

$$q_{1x}(t) = \left[-\frac{x}{(e-1)} e^{t/T} \left(1 + \frac{\theta_0}{2} t^2 - \theta_0 T t + \theta_0 T^2\right) + S \right.$$

$$\left. + \frac{x}{(e-1)} (1 + \theta_0 T^2) \right] e^{-\theta_0 t^2/2} \tag{8.19}$$

where S is the expected stock in hand in the beginning after satisfying the back orders.

Since there is no shortage, we have

$$q_{1x}(T) \geq 0$$

or $$\left[-\frac{x\,e}{(e-1)} \left(1 + \frac{\theta_0 T^2}{2}\right) + S + \frac{x}{(e-1)} (1 + \theta_0 T^2) \right] \geq 0$$

or $$S \geq x \left[1 + \frac{\theta_0 T^2 (e-2)}{2(e-1)} \right]$$

or $$S \geq \frac{x\,T}{2e} \left[\theta_0 T^2 (e-2) + 2(e-1) \right]$$

or $$x \leq \frac{S}{\left[1 + \frac{\theta_0 T^2 (e-2)}{2(e-1)} \right]} = S_1 \text{ (say)} \tag{8.20}$$

The average number of items carried in inventory per unit time

$$H_1(x) = \frac{1}{T} \int_0^T q_{1x}(t)\, dt, \qquad x \leq S_1$$

$$= \frac{x}{(e-1)} \left[-\frac{2}{3} e\, \theta_0 T^2 + 1 + 2\theta_0 T^2 \right] \tag{8.21}$$

The average number of items deteriorated per unit time

$$D_1(x) = \frac{1}{T} \left[S - x - q_{1x}(T \right]$$

$$= \frac{\theta_0 T}{2} \left[S - \frac{x}{(e-1)} \right] \tag{8.22}$$

and the average shortage per unit time

$$G_1(x) = 0 \tag{8.23}$$

Case II : When shortages occur.

In this case, differential equations describing the system are similar to equations (8.1) and (8.2) when d is replaced by x with their solutions as

$$q_{2x}(t) = \left[-\frac{x}{(e-1)} e^{t/T} \left(1 + \frac{\theta_0 T^2}{2} - \theta_0 T t + \theta_0 T^2 \right) + S + \frac{x}{(e-1)} (1 + \theta_0 T^2) \right] \times e^{-\theta_0 T^2/2}, \qquad 0 \le t \le t_1 \tag{8.24}$$

and $$q_{2x}(t) = \frac{x}{(e-1)} \left(e^{t_1/T} - e^{t/T} \right); \qquad t_1 < t \le T \tag{8.25}$$

Since shortage occur, we must have

$$q_{2x}(T) < 0$$

or $$x > S_1$$

where value of S_1 is given by expression (8.20). Also at $t = t_1$, $q_{2x} = 0$, which gives,

$$-\frac{x}{(e-1)} e^{t_1/T} \left(1 + \frac{\theta_0 t_1^2}{2} - \theta_0 T t_1 + \theta_0 T^2 \right) + \frac{x}{(e-1)} (1 + \theta_0 T^2) + S = 0$$

or $$e^{t_1/T} \left(1 + \frac{\theta_0 t_1^2}{2} - \theta_0 T t_1 + \theta_0 T^2 \right) = \frac{S(e-1)}{x} + (1 + \theta_0 T^2)$$

Taking log on both the sides, we have

$$\frac{t_1}{T} + \log \left\{ 1 + \theta_0 \left(T^2 + \frac{t_1^2}{2} - T t_1 \right) \right\} = \log \left\{ \frac{S(e-1)}{x} + (1 + \theta_0 T^2) \right\}$$

or $$\frac{t_1}{T} + \theta_0 T^2 + \frac{\theta_0 t_1^2}{2} - \theta_0 T\, t_1 = \log\left\{1 + \theta_0 T^2 + \frac{S\,(e-1)}{x}\right\} \qquad \text{as,} \qquad \theta << 1$$

or $$\frac{\theta_0 t_1^2}{2} + \left(\frac{1}{T} - \theta_0 T\right) t_1 + \theta_0 T^2 - \log\left\{1 + \theta_0 T^2 + \frac{S\,(e-1)}{x}\right\} = 0 \qquad (8.26)$$

which is a quadratic equation in t_1 and can be solved to get one positive root of t_1.

Now average number of items in inventory carried per unit time,

$$H_2(x) = \frac{1}{T}\int_0^{t_1} q_{2x}(t)\, dt$$

$$= \frac{x}{(e-1)}\left[e^{t_1/T}\left\{\frac{\theta_0 t_1^3}{3} - \theta_0 T\, t_1^2 + (1+2\theta_0 T^2)\, t_1 - T(1+2\theta_0 T^2)\right\} + T(1+2\theta_0 T^2)\right]$$

(8.27)

Average number of units deteriorating per unit time

$$D_2(x) = \frac{1}{T}\left[S - \frac{x}{(e-1)\,T}\int_0^{t_1} e^{t/T}\, dt\right]$$

$$= \frac{1}{T}\left[S - \frac{x}{(e-1)}\left(e^{t_1/T} - 1\right)\right] \qquad (8.28)$$

Also average shortage per unit time

$$G_2(x) = \frac{1}{T}\int_{t_1}^{T} q_{2x}(t)\, dt\,, \qquad x > S_1$$

$$= \frac{x}{(e-1)\,T}\left[2T\, e^{t_1/T} - t_1 e^{t_1/T} - e\,T\right] \qquad (8.29)$$

Therefore expected total cost per unit time of the system

$$K(t_1, S) = C\left[\int_0^{S_1} D_1(x)\, f(x)\, dx + \int_{S_1}^{\infty} D_2(x)\, f(x)\, dx\right] +$$

$$C_1\left[\int_0^{S_1} H_1(x)\, f(x)\, dx + \int_{S_1}^{\infty} H_2(x)\, f(x)\, dx\right]$$

$$- C_2\left[\int_0^{S_1} G_1(x)\, f(x)\, dx + \int_{S_1}^{\infty} G_2(x)\, f(x)\, dx\right] \tag{8.30}$$

where C is cost of unit item and C_1, C_2 are holding and shortage costs per unit per unit time respectively.

Substituting values of $D_1(x)$, $D_2(x)$, $H_1(x)$, $H_2(x)$, $G_1(x)$ and $G_2(x)$ and using value of t_1 obtained from equation (8.26) in equation (8.30), the total expected cost K(S) can be evaluated if probability density function f(x) is known.

The necessary condition for the total cost to be minimum is

$$\frac{d}{dS} K(S) = 0 \tag{8.31}$$

The total cost will be minimum is

$\frac{d^2}{dS^2} K(S) > 0$ at $S = S^*$ obtained from equation (8.31).

It can be verified that if we take $\theta \to 0$, i.e. there be no deterioration, all the expressions of this model reduce to the corresponding expressions of probabilistic demand model without deterioration derived in chapter seven.

DISCUSSION

In EOQ model presented in this chapter demand is a exponential function of time t and deterioration is variable, i.e. $\theta = \theta_0 t$, where θ_0 is a constant with $0 < \theta < 1$, $t > 0$. Production rate is infinite during the cycle period T and shortages are incarporated. The model is also discussed in case of probabilistic demand. Total average cost and total expected cost respectively have been obtained.

The above inventory model can further be generalized by taking finite rate of replenishments.

CHAPTER – 9

(T, S_i) POLICY INVENTORY MODEL FOR VARIABLE RATE OF DETERIORATION WITH LINEAR TREND IN DEMAND RATE

INTRODUCTION

One of the important problems confronting decision makers in modern organisation is to control and maintain inventories of deteriorating items, which is attacked by developing mathematical models of inventory dealing with deteriorating items under different circumstances. The term "deterioration" is well explained in chapter 4 and is now well evident in many inventory system and therefore impact of deterioration cannot be neglected. It were **Ghare and Sehrader** [41] who first developed an EOQ models for items with constant rate of deterioration, immediately thereafter followed by **Emmons** [38] with a similar exposition for the decay of radioactive nuclear generator. This work was extended by **Covert and Philip** [16] and **Philip** [81] by developing EOQ models for items with variable rate of deterioration. A further generalisation to all there is provided by **Shah** [94] by allowing shortages and considering general deterioration function. In all these models time is treated as a continuous variable. **Dave** [24, 25] produced inventory models considering time as a descrete variable.

Dave and Patel [24] presented a (T, S_i) policy inventory model for time proportional demand and in which items of inventory deteriorate at a constant rate. The above model was further extended by **Sachan** [98] by allowing shortages which are fully backlogged.

In the first section of this chapter, a (T, S_i) inventory model is

reconsidered for the situation of fixed cycle time with increasing (or decreasing) levels of order quantities and demand rate linearly changing with time. The model is formulated for variable rate of deterioration without shortages when planning horizon is finite and known and replenishment periods and assumed to be equal.

In the second section of this chapter, model is also discussed by permitting shortages. Finally it is shown that, if no deterioration occurs, the model developed is related to the corresponding model for non-deteriorating items as given by **Naddor** [75].

THE MODEL

SECTION – I

CASE WITHOUT SHORTAGES

The mathematical model of the (T, S_i) policy inventory system is developed under following assumptions :

(i) The system operates only for a prescribed period which is H units of time long.

(ii) During the period H there exists a total demand for d quantity units.

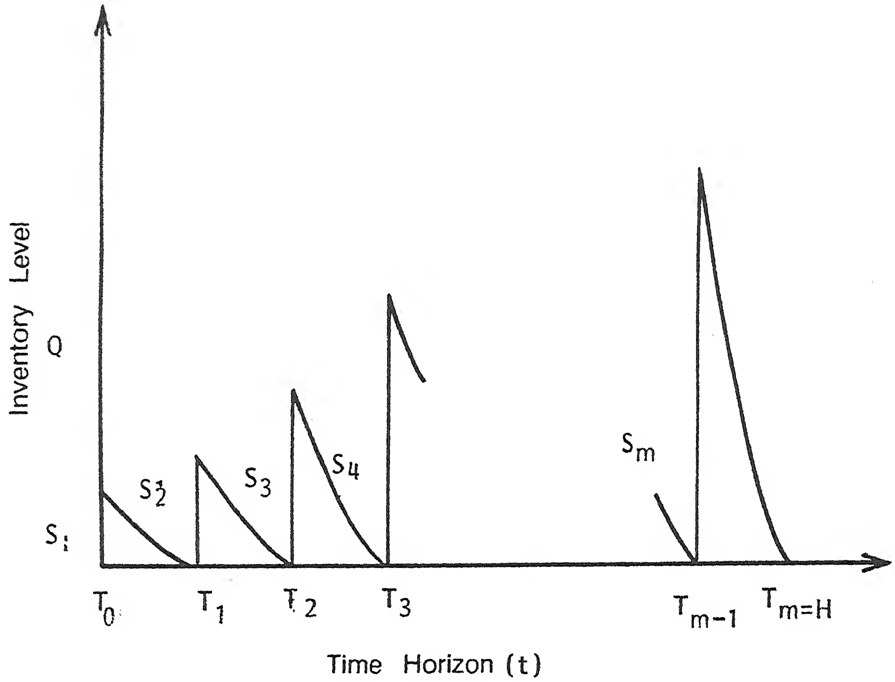
(iii) The demand rate r(t) is a linear function of time t, i.e.

$$r(t) = at, \qquad 0 \le t \le T \tag{9.1}$$

Then $d = \int_0^H r(t)\,dt = \frac{aH^2}{2}$, implies

$$r(t) = \frac{2d}{H^2}\,t, \qquad 0 \le t \le T \tag{9.2}$$

(iv) Let T_i be the total time that elapses upto and including the ith cycle, $i = 1, 2, \ldots m$, where m is the number of replenishments to be made during H. Also $T_0 = 0$ and $T_m = H$. The scheduling period is T, i.e., $T_i - T_{i-1} = T$ for all $i = 1, 2, \ldots m$. Then clearly



Graphical representation of (T, S_1) Policy Inventory Model without Shortages

$$T = \frac{H}{m} \tag{9.3}$$

and $$T_i = \frac{H}{m} i, \qquad i = 1, 2, \ldots, m \tag{9.4}$$

(v) Replenishment is instantaneous and lead time is zero. At the interval of every T time units, a variable quantity is ordered, so that the inventory level at the beginning of the ith cycle is equal to the order level S_i, $i = 1, 2, \ldots, m$. Since the cycle time is constant, we have $S_{i+1} > S_i$ (or $S_{i+1} < S_i$) when the demand is increasing (or decreasing).

(vi) Shortages are not allowed.

(vii) The unit cost C, the inventory holding cost C_1 per unit per unit time, C_2 is the shortage cost per unit per unit time and the replenishment cost C_3 per order are known and constant during the period under consideration.

(viii) A variable fraction $\theta(t)$ of the on hand inventory deteriorates per time unit with no repair or replacement. The functional form of deterioration is, as below

$\theta(t) = \theta_0 t$, where θ_0 is a constant and $0 < \theta_0 < 1$.

Figure on the left page shows the production cycle of the (T, S_i) model.

Let $q_i(t)$ denote the number of units in inventory at any time t during ith cycle; $T_{i-1} \le t \le T_i$, $i = 1, 2, \ldots, m$. Then differential equation governing the ith cycle of the system is

$$\frac{d}{dt} q_i(t) + \theta_0 t\, q_i(t) = -\frac{2d}{H^2} t; \qquad T_{i-1} \le t \le T_i \tag{9.5}$$

Solution to which is given as,

$$q_i(t)\, e^{\theta_0 t^2/2} = -\frac{2d}{H^2} \int e^{\theta_0 t^2/2} t\, dt + \alpha,$$

where $e^{\theta_0 t^2/2}$ is the integrating factor and α is constant of integration.

or $$q_i(t)\, e^{\theta_0 t^2} = -\frac{2d}{\theta_0 H^2} e^{\theta_0 t^2/2} + \alpha$$

Since $q_i(T_i) = 0$, above expression gives,

$$\alpha = \frac{2d}{\theta_0 H^2} e^{\theta T_i^2/2}$$

Therefore solution to differential equation (9.5) becomes,

$$q_i(t) = \frac{2d}{\theta_0 H^2}\left[\exp\left\{\frac{\theta_0}{2}(T_i^2 - t^2)\right\} - 1\right] \tag{9.6}$$

$$T_{i-1} < t < T_i, \quad i = 1, 2, \ldots, m$$

Also at $t = T_{i-1}$, order-level is S_i, so

$$S_i = \frac{2d}{\theta_0 H^2}\left[\exp\left\{\frac{\theta_0}{2}(T_i^2 - T_{i-1}^2)\right\} - 1\right] \tag{9.7}$$

The total demand during ith cycle is

$$\int_{T_{i-1}}^{T} r(t)\, dt = \frac{2d}{H^2}\left[\frac{t^2}{2}\right]_{T_{i-1}}^{T_i}$$

$$= \frac{d}{H^2}\left[T_i^2 - T_{i-1}^2\right]$$

$$= \frac{d}{H^2}\left[\frac{H^2}{m^2} i^2 - \frac{H^2}{m^2}(i-1)^2\right]$$

$$= \frac{d}{m^2}\left[i^2 - (i-1)^2\right]$$

$$= \frac{d}{m^2}(2i - 1) \tag{9.8}$$

Hence number of deteriorated units during the ith cycle,

$$D_i = S_i - \frac{d}{m^2}(2i - 1)$$

$$= \frac{2d}{\theta_0 H^2}\left[\exp\left\{\frac{\theta_0}{2}\left(T_i^2 - T_{i-1}^2\right)\right\} - 1\right] - \frac{d}{m^2}(2i-1)$$

$$i = 1,\ 2,\ \ldots,\ m$$

$$= \frac{2d}{\theta_0 H^2}\left[\frac{\theta_0}{2}\left(T_i^2 - T_{i-1}^2\right) + \frac{\theta_0^2}{8}\left(T_i^2 - T_{i-1}^2\right)\right] - \frac{d}{m^2}(2i-1)$$

$$= \frac{d}{H^2}\left[(T_i^2 - T_{i-1}^2) + \frac{\theta_0}{4}(T_i^4 - 2T_i^2T_{i-1}^2 + T_{i-1}^4)\right] - \frac{d}{m^2}(2i-1)$$

$$= \frac{d}{H^2}\left[\frac{H^2}{m^2}(2i-1) + \frac{\theta_0 H^4}{4m^4}\left\{i^4 - 2\,i^2(i-1)^2 + i^4\right\}\right] - \frac{d}{m^2}(2i-1)$$

$$= \frac{d\theta_0 H^2}{4m^4}(2i-1)^2 \tag{9.9}$$

Hence during H, total number of deteriorated items

$$D(m) = \sum_{i=1}^{m} D_i$$

$$= \frac{d\theta_0 H^2}{4m^4}\sum_{i=1}^{m} D_i$$

$$= \frac{d\theta_0 H^2}{4m^4}\left[\frac{2m\,(m+1)\,(2m+1)}{3} - 2m\,(m+1) + m\right]$$

$$= \frac{d\theta_0 H^2}{6m^3}(2m^2 - 1) \tag{9.10}$$

Again number of units in inventory during ith cycle,

$$I_i = \int_{T_{i-1}}^{T_i} q_i(t)\,dt$$

$$= \frac{2d}{\theta_0 H^2}\int_{T_{i-1}}^{T_i}\left[\exp\left\{\frac{\theta_0}{2}(T_i^2 - t^2)\right\} - 1\right]dt$$

$$= \frac{2d}{\theta_0 H^2} \int_{T_{i-1}}^{T_i} \left[\frac{\theta_0}{2}(T_i^2 - t^2) + \frac{\theta_0^2}{8}(T_i^2 - t^2)^2\right] dt$$

$$= \frac{d}{H^2} \int_{T_{i-1}}^{T_i} \left[(T_i^2 - t^2) + \frac{\theta_0}{4}(T_i^4 - 2T_i^2 t^2 + t^4)\right] dt$$

$$= \frac{d}{H^2} \left[\left(T_i^2 t - \frac{t^3}{3}\right) + \frac{\theta_0}{4}\left(T_i^4 t - \frac{2T_i^2}{3} t^3 + \frac{t^5}{5}\right)\right]_{T_{i-1}}^{T_i}$$

$$= \frac{d}{H^2} \left[\left(\frac{2}{3}T_i^3 - T_i^2 T_{i-1} + \frac{T_{i-1}^3}{3}\right) + \frac{\theta_0}{4}\left(\frac{8}{15}T_i^5 - T_i^4 T_{i-1} + \frac{2}{3}T_i^2 T_{i-1}^3 - \frac{T_{i-1}^5}{5}\right)\right]$$

$$= \frac{dH}{m^3}\left[\left\{\frac{2}{3}i^3 - i^2(i-1) + \left(\frac{i-1}{3}\right)^3\right\} + \frac{\theta_0 H^2}{4m^2}\left\{\frac{8}{15}i^5 - i^4(i-1) + \frac{2}{3}i^2(i-1)^3 - \frac{(i-1)^5}{5}\right\}\right]$$

$$= \frac{dH}{3m^3}\left[(3i-1) + \frac{\theta_0 H^2}{20} m^2 (20i^2 - 15i + 3)\right]$$

$$= \frac{dH}{3m^3}\left[\frac{\theta_0 H^2}{m^2} i^2 - 3\left(\frac{\theta_0 H^2}{4m^2} - 1\right) i + \left(\frac{3\theta_0 H^2}{20} m^2 - 1\right)\right] \tag{9.11}$$

Therefore total number of units in inventory during the time H,

$$I(m) = \sum_{i=1}^{m} I_i$$

$$= \frac{dH}{3m^3}\left[\frac{\theta_0 H^2}{m^2} \Sigma\, i^2 - 3\left(\frac{\theta_0 H^2}{4m^2} - 1\right) \Sigma\, i + \left(\frac{3\theta_0 H^2}{20m^2} - 1\right) \Sigma\, 1\right]$$

$i = 1, 2 \ldots, m.$

$$= \frac{dH}{3m^2}\left[\frac{\theta_0 H^2}{6m^2}(2m^2 + 3m + 1) - \frac{3}{2}\left(\frac{\theta_0 H^2}{4m^2} - 1\right)(m+1) + \left(\frac{3\theta_0 H^2}{20m^2} - 1\right)\right]$$

$$= \frac{dH}{3m^2}\left[\frac{3}{2}m + \left(\frac{1}{2} + \frac{\theta_0 H^2}{3}\right) + \frac{\theta_0 H^2}{8m} - \frac{7}{120}\frac{\theta_0 H^2}{m^2}\right] \tag{9.12}$$

Finally the total average cost per time unit of the system for period H is given by

$$K(m) = \frac{C}{H} D(m) + \frac{C_1}{H} I(m) + \frac{C_3}{H} m \tag{9.13}$$

$$= \frac{d\theta_0 H}{6m^3} C(2m^2+1) + \frac{d}{3m^2} C_1 \left[\frac{3}{2} m + \left(\frac{1}{2} + \frac{\theta_0 H^2}{3}\right) + \frac{\theta_0 H^2}{8m} - \frac{7}{120}\frac{\theta_0 H^2}{m^2}\right] + \frac{m}{H} C_3 \tag{9.14}$$

Since m is a positive integer, the necessary condition for the total average cost to be minimum at $m = m_0$ is

$$\Delta K(m_0 - 1) \le 0 \le \Delta K(m_0) \tag{9.15}$$

where $\Delta K(m) = K(m + 1) - K(m)$ (9.16)

Using cost expression (9.14) in condition (9.15) for optimality of m at m_0 with the help of equation (9.16), we obtain,

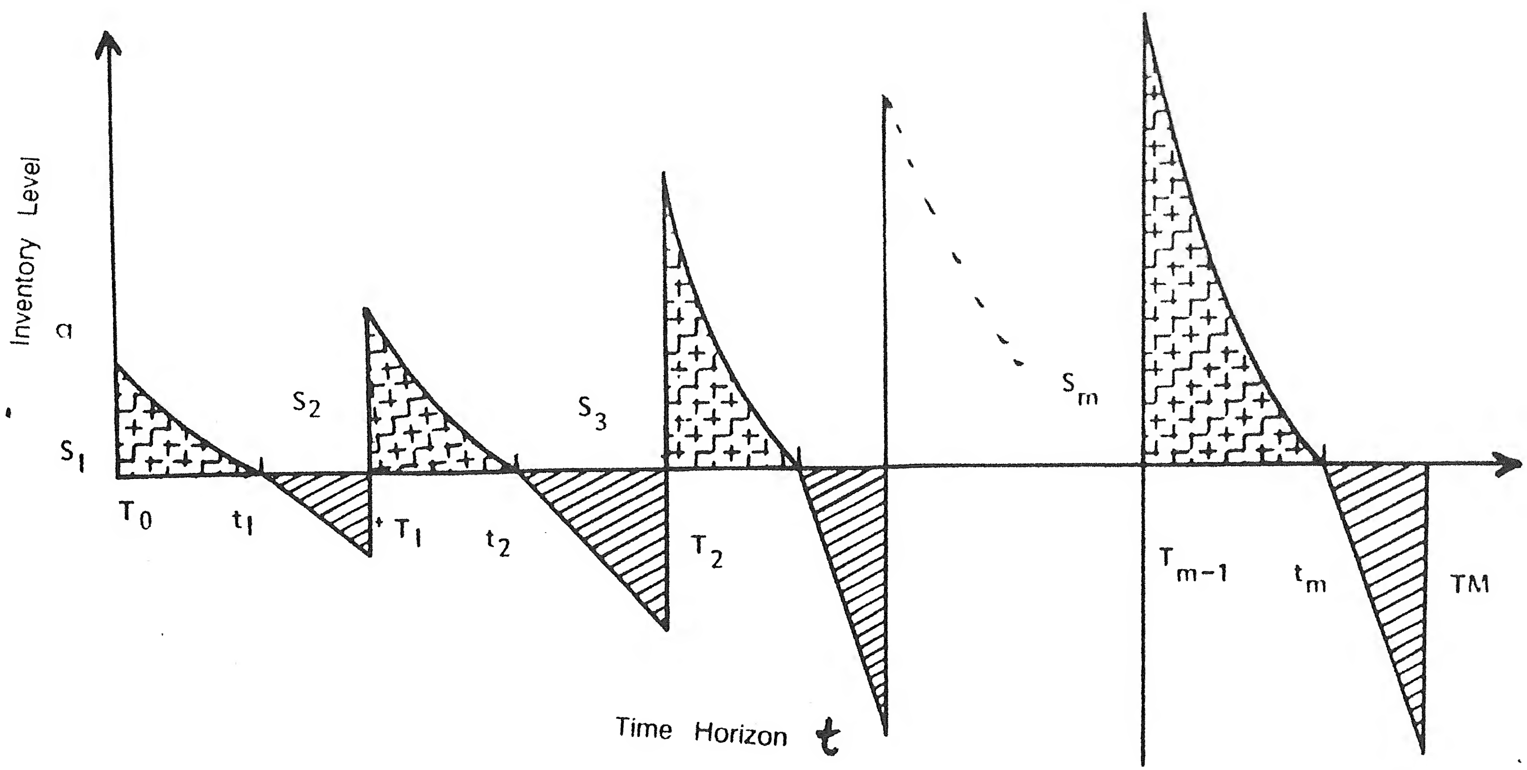
$$P(m_0) \le \frac{C_3}{Hd} \le P(m_0 - 1) \quad \text{with} \tag{9.17}$$

$$P(m) = -[\Delta P_1(m) + \Delta P_2(m)] \tag{9.18}$$

where $P_1(m) = \dfrac{C\theta_0 H}{6m^3}(2m^2 + 1)$ (9.19)

and $P_2(m) = \dfrac{C_1}{3m^2}\left[\dfrac{3}{2} m + \left(\dfrac{1}{2} + \dfrac{\theta_0 H^2}{3}\right) + \dfrac{\theta_0 H^2}{8m} - \dfrac{7}{120}\dfrac{\theta_0 H^2}{m^2}\right]$ (9.20)

Substituting value of $m = m_0$ as obtained from expression (9.17) in expressions (9.3), (9.7) and (9.14), the optimal scheuling period T_0, the optimal order levels S_{i_0}, $i = 1, 2, \ldots, m$ and the minimum total cost $K(m_0)$ of the system can be obtained.



Graphical representation of (T, S_1) Policy Inventory Model with Shortages

In particular if $\theta = 0$, expressions (9.7), (9.14) and (9.17) reduces to

$$S_i = \frac{d}{m^2}(2i - 1) \tag{9.21}$$

$$K(m) = \frac{dC_1}{6m^2}(3m + 1) + \frac{m}{H}C_3 \tag{9.22}$$

and $$R(m_0 - 1) \le \frac{C_1 dH}{C_3} \le R(m_0) \tag{9.23}$$

where $$R(m) = \frac{6m^2(m+1)^2}{(3m^2 + 5m + 1)} \tag{9.24}$$

Equations (9.21), (9.22), (9.23) and (9.24) are the same as those given by **Naddor** [75] for the (t, S_i) policcy inventory model for non-deteriorating items.

SECTION – II

CASE WITH SHORTAGES

In this situation all the assumptions and conditions are same as in inventory model without shortages derived in this chapter except that shortages are allowed here and t_i is the total time that elapses upto and incluuding the $(i - 1)$th cycle and the consumption period of ith cycle, $i = 1, 2, \ldots, m$. Adjacent figure shows that production cycle of model.

Also $$t_i - T_{i-1} = r(T_i - T_{i-1}); \quad i = 1, 2, \ldots, m \tag{9.25}$$

Where r is a fraction such that $0 \le r \le 1$. It follows from equation (9.25) that when $r = 1$, shortages are not allowed and when $r = 0$, the inventories are not carried at all during the planning period H.

Equation (9.25) can be written as

$$t_i = (1 - r)T_{i-1} + rT_i$$

$$= \frac{H}{m}\left[(1-r)(i-1)+ir\right] \quad \text{from equation (9.4).}$$

$$= \frac{H}{m}(i-1+r) \tag{9.26}$$

Differential equations describing the ith cycle of the system, in this case are as follows,

$$\frac{d\,(q_i t)}{dt} + \theta_0 t\, q_i(t) = -\frac{2dt}{H^2}, \qquad T_{i-1} \le t \le t_i \tag{9.27}$$

$$i = 1,\ 2,\ \ldots,\ m$$

and

$$\frac{d}{dt} q_i(t) = -\frac{2d}{H^2} t, \qquad t_i \le t \le T_i \tag{9.28}$$

$$i = 1,\ 2,\ \ldots,\ m$$

where $q_i(t)$ is the current stock level at any time t. Solutions to which are given by

$$q_i(t) = \frac{2d}{\theta_0 H^2}\left[\exp\left\{\frac{\theta_0}{2}(t_i^2 - t^2) - 1\right\}\right] \qquad T_{i-1} \le t \le t_i \tag{9.29}$$

and

$$q_i(t) = \frac{d}{H^2}(t_i^2 - t^2), \qquad t_i \le t \le T_i \tag{9.30}$$

Since initially at ith cycle, inventory level is S_i i.e.. at $t = T_{i-1}$, $q_i(t) = S_i$, we get

$$S_i = \frac{2d}{\theta_0 H^2}\left[\exp\left\{\frac{\theta_0}{2}(t_i^2 - T_{i-1}^2)\right\} - 1\right]$$

$$= \frac{2d}{\theta_0 H^2}\left[\exp \frac{\theta_0 H^2}{2m^2}\left\{(i-1+r)^2 - (i-1)^2\right\} - 1\right] \quad \text{using equation (9.26).}$$

$$= \frac{2d}{\theta_0 H^2}\left[\exp \frac{\theta_0 H^2}{2m^2}(r^2 + 2ir - 2r) - 1\right] \tag{9.31}$$

Total number of units that deteriorate during the ith cycle is,

$$D_i = S_i - \frac{2d}{H^2} \int_{T_{i-1}}^{t} t\,dt$$

$$= \frac{2d}{\theta_0 H^2}\left[\exp \frac{\theta_0 H^2}{2m^2}(r^2 + 2ir - 2r) - 1\right] - \frac{d}{H^2}\left[t_i^2 - T_{i-1}^2\right]$$

$$= \frac{2d}{\theta_0 H^2}\left[\frac{\theta_0 H^2}{2m^2}(r^2 + 2ir - 2r) + \frac{\theta_0^2 H^4}{8m^4}(r^2 + 2ir - 2r)^2\right] - \frac{d}{m^2}(r^2 + 2ir - 2r)$$

Neglecting higher order terms of θ_0 as $0 < \theta < 1$.

$$= \frac{d\theta_0 H^2}{4m^4}\left[r^4 + 4r^2(i-1)^2 + 4r^3(i-1)\right] \tag{9.32}$$

Hence total number of units deteriorating during H, is

$$D(m) = \sum_{i=1}^{m} D_i$$

$$= \frac{d\theta_0 H^2}{4m^4}\left[4r^2 \sum_{i=1}^{m}(i-1)^2 + 4r^3 \sum_{i=1}^{m}(i-1) + r^4 \sum_{i=1}^{m} 1\right]$$

$$= \frac{d\theta_0 H^2}{4m^4}\left[\frac{2}{3}r^2 m(m-1)(2m-1) + 2r^3 m(m-1) + r^4 m\right]$$

$$= \frac{d\theta_0 H^2 r^2}{12m^3}\left[2m(m-1)(2m-1) + 6r(m-1) + 3r^2\right] \tag{9.33}$$

Number of units in inventory in ith cycle

$$I_1(i) = \int_{T_{i-1}}^{t_i} q_i(t)\,dt$$

$$= \frac{d}{H^2}\int_{T_{i-1}}^{t_i}\left[\left(t_i^2 - t^2\right) + \frac{\theta_0}{4}(t_i^2 - t^2)^2\right] dt$$

$$= \frac{d}{H^2}\left[\left(t_i^2 t - \frac{t^3}{3}\right) + \frac{\theta_0}{4}\left(t_i^4 t - \frac{2}{3}t_i^2 t^3 + \frac{t^5}{5}\right)\right]_{T_{i-1}}^{t_i}$$

$$= \frac{d}{H^2}\left[\frac{2}{3}t_i^3 - \left(t_i^2 T_{i-1} - \frac{T_{i-1}^3}{3}\right) + \frac{2\theta_0 t_i^5}{15} - \frac{\theta_0}{4}\left(t_i^4 T_{i-1} - \frac{2}{3}t_i^2 T_{i-1} + \frac{T_{i-1}^5}{5}\right)\right]$$

$$= \frac{d}{H^2}\left[\frac{2\theta_0 H^5}{15m^5}(i-1+r)^5 - \frac{\theta_0 H^5}{4m^5}\left\{(i-1+r)^4(i-1) - \frac{2}{3}(i-1+r)^2(i-1)^3 + \frac{1}{5}(i-1)^5\right\}\right.$$

$$\left. + \frac{2H^3}{2m^3}(i-1+r)^3 - \frac{H^3}{m^3}\left\{(i-1+r)^2(i-1) - \frac{(i-1)^3}{3}\right\}\right] \qquad (9.34)$$

Hence total number of units in inventory during H,

$$I_1(m) = \sum_{i=1}^{m} I_1(i)$$

$$= \frac{dH}{m^3}\left[\frac{\theta_0 H^2 r^3}{3m^2}\sum_{i=1}^{m}(i-1)^2 + \left(r^2 + \frac{5}{12m^2}\theta_0 H^2 r^4\right)\sum_{i=1}^{m}(i-1)\right.$$

$$\left. + \frac{2}{3}r^3\left(1 + \frac{\theta_0 H^2 r^2}{5m^2}\right)\sum_{i=1}^{m}1\right]$$

$$= \frac{dHr^2}{m^2}\left[\frac{\theta_0 H^2 r}{18m^2}(m-1)(2m-1) + \frac{1}{2}\left(1 + \frac{5\theta_0 H^2 r^2}{12m^2}\right)(m-1) + \frac{2r}{3}\left(1 + \frac{\theta_0 H^2 r^2}{5m^2}\right)\right]$$

$$= \frac{dHr^2}{m^2}\left[\frac{1}{2}m + \left(\frac{\theta_0 H^2 r}{9} + \frac{2}{3}r - \frac{1}{2}\right) + \left(\frac{5\theta_0 H^2 r^2}{24} - \frac{\theta_0 H^2 r}{6}\right)\frac{1}{m} + \left(\frac{\theta_0 H^2 r}{18} - \frac{27\theta_0 H^2 r^2}{120}\right)\frac{1}{m^2}\right]$$

(9.35)

Also number of units responsible for shortages during ith cycle

$$I_2(i) = \int_{t_i}^{T_i} q_i(t)\, dt$$

$$= \frac{d}{H^2} \int_{t}^{T} (t_i^2 - t^2)\, dt$$

$$= \frac{d}{H^2} \left[\left(t_i^2 t - \frac{t^3}{3} \right) \right]_{t}^{T}$$

$$= \frac{d}{H^2} \left[t_i^2 T_i - \frac{T_i^3}{3} - \frac{2}{3} t_i^3 \right]$$

$$= \frac{dH}{3m^3} \left[3(i - 1 + r)\, i - i^3 - 2(i - 1 + r)^3 \right]$$

$$= \frac{dH}{3m^3} (1 - r)^2 (3i - 2 + 2r) \qquad (9.36)$$

Therefore total number of units backloggged during H,

$$I_2(m) = \frac{d}{6m^2} (1 - r)^2 (3m - 1 + 4r) \qquad (9.37)$$

Now total average cost per unit per unit of time

$$K(m, r) = \frac{d\theta_0 H r^2 C}{12m^3} \left[2(m - 1)(2m - 1) + 6r(m - 1) + 3r^2 \right]$$

$$+ \frac{dr^2}{m^2} C_1 \left[\frac{m}{2} + \left(\frac{\theta_0 H^2 r}{9} + \frac{2}{3} r - \frac{1}{2} \right) + \left(\frac{5\theta_0 H^2 r^2}{24} - \frac{\theta_0 H^2 r}{6} \right) \frac{1}{m} \right.$$

$$\left. + \left(\frac{\theta_0 H^2 r}{18} - \frac{27\theta_0 H^2 r^2}{120} \right) \frac{1}{m^2} \right] + \frac{dC_2}{6m^2} \left[(1 - r)^2 (3m + 4r - 1) \right] + \frac{C_3}{H} m$$

(9.38)

Since m is a non-negative integer and $r \in [0, 1]$, the necessary conditions for $K(m, r)$ to be minimum at $m = m_0$ and $r = r_0$ are

$$\Delta K(m_0 - 1, r) \le 0 \le \Delta K(m_0, r) \qquad (9.39)$$

and $\frac{\partial}{\partial r} K(m, r) = 0$ at $r = r_0$ (9.40)

where $\Delta K(m) = K(m+1) - K(m)$ (9.41)

By virtue of equation (9.38) and (9.41) condition for minimum cost reduce,

$$P(m_0) \le \frac{C_3}{Hd} \le P(m_0 - 1) \tag{9.42}$$

with $P(m) = -[\Delta P_1(m) + \Delta P_2(m) + \Delta p_3(m)]$ (9.43)

where $P_1(m) = \frac{\theta_0 Hr^2 C}{12m^3}\left[2(m-1)(2m-1) + 6r(m-1) + 3r^2\right]$ (9.44)

$$P_2(m) = \frac{r^2 C_1}{m^2}\left[\frac{m}{2} + \left(\frac{\theta_0 H^2 r}{9} + \frac{2}{3}r - \frac{1}{2}\right) + \left(\frac{5\theta_0 H^2 r^2}{24} - \frac{\theta_0 H^2 r}{6}\right)\frac{1}{m}\right.$$

$$\left. + \left(\frac{\theta_0 H^2 r}{18} - \frac{27\theta_0 H^2 r^2}{120}\right)\frac{1}{m^2}\right] \tag{9.45}$$

$$P_3(m) = \frac{C_2}{6m^2}\left[(1-r)^2(3m + 4r - 1)\right] \tag{9.46}$$

and,

$$\frac{d\theta_0 HC}{6m^3}\left[2r(m-1)(2m-1) + 3(m-1) + 3r\right]$$

$$+ \frac{d}{m^2}C_1\left[rm + \left(\frac{\theta_0 H^2 r^2}{3} + 2r^2\right) + \left(\frac{5\theta_0 H^2 r^3}{6} - \frac{\theta_0 H^2 r^2}{2}\right)\frac{1}{m}\right.$$

$$\left. + \left(\frac{\theta_0 H^2 r^2}{6} - \frac{27\theta_0 H^2 r^3}{30}\right)\frac{1}{m^2}\right] - \frac{4}{3}\frac{dC_2}{m^2}(1-r) = 0 \tag{9.47}$$

Equations (9.42) and (9.47) are optimal conditions.

In particular when there is no deterioration, we have

$$\lim_{\theta \to 0} K(m, r) = G(m, r) = \frac{dr^2 C_1}{6m^2}(3m+4r-3) + \frac{dC_2}{6m^2}(1-r)^2(3m+4r-1) + \frac{C_3 m}{H} \tag{9.48}$$

$$\text{Also} \quad G(m, 1) = \frac{dC_1}{6m^2}(3m + 1) + \frac{C_3 m}{H} \tag{9.49}$$

$$\text{and} \quad G(m, 0) = \frac{dC_2}{6m^2}(3m - 1) + \frac{C_3 m}{H} \tag{9.50}$$

Equation (9.49) is the same as obtained by equation (9.22) for no shortage case given by **Naddor** [75]. Solving equations (9.42) and (9.47) the optimal values m_0 and r_0 can be obtained. Substituting these values of m_0 and r_0 in equation (9.31) and (9.38) the optimal order levels S_i and minimum average cost $C(m_0, r_0)$ of the system can be obtained.

The demand of a product may decrease with time due to the introduction of a new product which is cheaper and superior to old one. The value of new product will be increasing with time. Most of the food products. photographic films, many drugs and pharmaceuticals will fall into any one of these two categories. The models studied in this chapter can be useful for all such products which show increasing or decreasing trend in their demand. For example fuel consumption during winter, petrol consumption during vacation etc.

DISCUSSION

A (T, S_i) policy inventoty model has been investigated in this chapter for linear trend in demand and variable rate of deterioration. In first part of the chapter, shortages are not allowed while in second part shortages are also incarporated. The number of replenishments period, scheduling time and total cost are, all determined in an optimal manner.

It is deduced that, if deterioration does not occur, both of these models reduce to the corresponding model for non-deteriorating items, given by **Naddor** [75].

The above (T, S_i) policy inventory model can further be generalized for other suitable functional forms of demand rate.

REFERENCES

1. AGGARWAL, S. P. (1978),
   A note on an order-level inventory model for a system with constant rate of deterioration.
   Opsearch, 15, 184-187.

2. AGGRAWAL, S.C. (1981),
   Purchase inventory decision models for inflationary conditions.
   Interface 11, 18.

3. AUCAMP, D.C. and KUZDRALL, P.J. (1986),
   Lot sizes for one time sales,
   J. opl. Res., 37, 79-86.

4. ALTIOK, TAYFUR (1987)
   (R, r) Production/Inventory Systems.
   Operations Research, 266-276.

5. ATTILA CHIKAN (1991)
   Inventory Models, (ed.),
   Kluwer Academic Publishers, Dordrecht. The Netherlands.

6. BARANKIN, E. and DENNY, J. (1960)
   Examination of an inventory model incarporating probabitites of obsolescence.
   Technical Report No 18, *Statistical Laboratory*, University of California, Berkeley, California..

7. BROWN, G., LU, J. AND WOLFSON, R. (1964),
   Dynamic modelling of inventory subject to obsolescence.
   Management Science, 11, 51-63.

8. BULINSKAYA, E. (1964)
   Some results concerning optimal inventory policies.
   Theory of prob. and its appl., 9, 386-403.

9. BUCHANAN, J. E. (1980)
   Alternative solution methods for the inventory replenishment problem under increasing demand.

J. Opl. Res. Soc. 31, 615-620.

10. BACKER, R.C. and VILCASSIM, N. (1983)
    Continuous review price change inventory model.
    Prod. Invent. Mgmt. 24, 55-64.

11. BENERJEE, A. (1986)
    A joint economic lot size model for purchaser and vendor.
    Decision Sciences, 17, 292-311.

12. BACKER, R.C. and HANNA MICHAEL (1987)
    Cost comparision for out of phase inventory models.
    J. opl. Res. Soc. 38 (3), 253-260.

13. BAKER, R.C. and URBAN, T.L. (1988)
    A deterministic inventory stysem with an inventory level dependent demand rate.
    J. opl. Res. Soc., 39, 823-831.

14. BAHARI KASANI, H. (1989)
    Replenishment schedule for determinating items with time propotional demand.
    J. opl. Res. Soc., 40, 75-81.

15. CROSTON, J.D. (1972)
    Forecasting and stock control for intermittent demand.
    Opl. Res. Q., 23, 289-303.

16. COVERT, R.P. and PHILIP G.C. (1973)
    An EOQ model for items with Weibull distribution deterioration.
    AIIE Tran., 5, 323-329.

17. COHEN, M.A. (1977)
    Joint pricing and ordering policy for exponentially decaying inventory with known demand.
    Nav. Res. Log. Quart., 24 (2), 257-268.

18. CHAKRAVARTY, A.K. (1981)
    Multi-item inventory aggregation into groups.
    J. opl. Res. Soc., 32, 19-26.

19. CHOWDHURY, MITALI ROY and CHAUDHURI, K.S. (1983)
An order level inventory model for deteriorating items with finite rate of replenishment.
Opsearch

20. CHAPMAN, C.B., WARD, S.C., COOPER, D.F. and PAGE, M.J. (1984)
Credit policy and inventory control.
J. opl Res. Soc 35, 1053-1065.

21. CHAPMAN, C.B. and WARD, S.C. (1988)
Inventory control and trade credit – a further reply.
J. opl Res. Soc., 39, 219-220.

22. CHATTERJEE, A.K. and RAVI, R. (1991)
Joint economic lot size model with delivery in sub-batches.
Opsearch, 28 (2), 118-124.

23. DONALDSON, W.A. (1977)
Inventory replenishment policy for a linear trend in demand, an analytic solution.
Opl. Res. Q., 28, 663-670.

24. DAVE, U. (1978)
A discrete in time EOQ model for deteriorating items.
GUJ. statist. Rev., 5, 51-56.

25. DAVE, U. (1979)
On a discrete in time order level inventory model for deteriorating items.
An. Inst. Ind. Engng Trans. 9, 108-112.

26. DAVE, U. and PATEL, L.K. (1981)
(T, S_i) Policy inventory model for deteriorating items with time propotional demand.
J. opl. Res. Soc., 32, 137-142.

27. DAVE, UPENDRA (1986)
An order level inventory model for deteriorating items with variable instantaneous demand and discrete opportunities for replenishment.
Op search, 23 (4), 244-249.

28. DEB, M. and CHAUDHURI, K.S. (1986),
An EOQ model for items with finite rate of production and variable rate of deterioration.
Opsearch, 23, 175-181.

29. DAELLENBACH, H. G. (1986)
Inventory control and trade credit.
J. opl. Res. Soc., 37, 525-528.

30. DEB, M and CHAUDHURI K.S. (1987)
A note on the heuristic for replenishment of trended inventories considering shortages.
J. opl. Res. Soc., 38, 459-463.

31. DAVE, UPENDRA (1988)
On the EOQ models with two levels of shortage.
Opsearch, 25 (3), 190-196.

32. DATTA, T.K. and PAL, A.K. (1988)
Order level inventory with power demand pattern for items with variable rate of deterioration.
Indian J. pure appl. Math., 19 (11), 1043-1053.

33. DAELLENBACH, H.G. (1988)
Inventory control and trade credit – a further reply.
J. Opl. Res. Soc., 39, 219-220.

34. DAVE, U. (1989)
On a heuristic inventory replenishment rule for items with a lenearly increasing demand incarporating shortages.
J. Opl. Res. Soc., 40, 827-830.

35. DAVE, U. (1989)
A deterministic lot size inventory model with shortages and a linear trend in demand.
Naval Res. Logist, 36, 507-514.

36. DATTA, T.K. and PAL, A.K. (1990)
A note on an inventory model with inventory level dependent demand rate.

J. opl. Res. Soc., 41 (10), 971-975.

37. DATTA, T.K. and PAL, A.K. (1992)
A note on replenishment policy for an inventory model with linear trend in demand and shortages.
J. opl. Res. Soc., 44, 993-1001.

38. EMMONS, H. (1968),
The replenishment or directly used radioactive pharmaceuticals in medical diagonosis, unpublished doctoral dissertation.
Johns Hopkin University, Baltimore, Maryland.

39. Eck, R.W. (1976)
Operations Research for business.
Wadsworth, London, p. 395.

40. FRIES, B.E. (1975)
Optimal ordering policy for a perishable commodity with fixed life time.
Operations Research, 23, 46-61.

41. GHARE, P.M. and SCHRADER, G.F. (1963)
A model for exponentially decaying inventories.
Jr. Ind. Engng., 14, 238-243.

42. GROSS, D. and HARRIS, C.M. (1973)
Continuous review (s, S) inventory models with state dependent lead times.
Mgmt. Sci., 19, 567-574.

43. GOYAL, S.K. (1973)
Scheduling a multi-product single machine system.
Opl. Res. Q., 24, 261-269.

44. GOYAL, S.K. (1977)
Determination of optimum production quantity for a stage production system.
Opl. Res. Q., 28, 865-870.

45. GOYAL, S.K. (1979)
A note on the paper: An inventory model with finite horizon and price

changes.
J. Opl. Res. Soc., 30, 839-840.

46. GAVISH, B. and GRAVES, S.C. (1980)
A one product production/inventory problem under continuous review policy.
Opns. Res., 28, 1228-1236.

47. GOYAL, V.P. and AGGRAWAL S.P. (1981)
Proceedings All India Seminar on Operational Research and Decision Making.

48. GOSWAMI, ADRIJIT and CHAUDHURI, K.S. (1981)
An EOQ model for deteriorating items with shortages and linear trend in demand.
J. Opl. Res. Soc., 42 (12), 1105-1110.

49. GURNANI, C. (1983)
Economic analysis of inventory systems.
Int. J. Prod. Res., 21, 261-277.

50. GUPTA, R. and VRAT, P. (1988),
Inventory model for stock dependent consumption rate.
Opesearch, 23, 19-24.

51. GOYAL, S.K. and BHATT, S.K. (1988)
A generalized lot size ordering policy for price increases.
Opsearch, 25 (4), 272-278.

52. GOYAL, S.K. (1990),
A note on inventory model with cost increases.
J. opl. Res., 414-415.

53. GOSWAMI, A. and CHAUDHURI, K. (1991),
EOQ model for an inventory with a linear trend in demand and finite rate of replenishment considering shortages.
Int. J. Syst. Sci., 22, 181-187.

54. GOSWAMI, ADRIJIT and CHAUDHURI, K.S. (1992),
An economic order quantity model for items with two level of storage

for linear trend in demand.
J. Opl. Res. Soc., 43 (2), 157-167.

55. HADLEY, G. and WHITIN, T.M. (1961)
An optimal final inventory model.
Management Science, 7, 179-183.

56. HADLEY, G. and WHITIN, T.M. (1962)
Generalization of the optimal final inventory model.
Management Science, 8, 454-458.

57. HADLEY, G. and WHITIN, T.M. (1963)
Analysis of Inventory System
Prentice Hall, Englewood, Cliff, J.

58. HARTELY, R.V. (1976)
Managerial Emphasis.
Good year publishing company, California 315-317.

59. HARI KISHAN and MISHRA, P.N. (1995)
Inventory model with stock dependent demand rate and constant rate of deterioration.
Acta Ciencia Indica, Vol.XXI (4M) 420-422.

60. KARMARKAR, U.S. (1987)
Lot sizes, lead time and in-process inventories.
Management Science, 33, 409-418.

61. KRISHNAMOORTHY, A. and MANOHARAN, M. (1991)
An inventory system with unit demand and varying ordering levels.
Optimization, 22 (2), 283-289.

62. LOVE, STEPHEN; F. (1972)
A facilities in series inventory model with nested sehedules.
Management Science, 18 (5), 327-328.

63. LEY, B. and SOYSTER A.L. (1979)
An inventory model with finite horizon and price changes.
J. Opl. Res. Soc., 30, 43--53.

64. LUSS, HANAN and ROSENWEIN, B. MOSHE (1990)
A lot sizing model for just in time manufacturing.
J. Opl. Res. Soc., 41 (3), 201-209.

65. MURRAY, Jr. G.R. and SILVER, E.R. (1966)
A Bayesian analysis of the style goods inventory problems.
Management Science, 12, 785-797.

66. MAGEE, J.F. and BOODMAN, D.M. (1967)
Production, Planning and Inventory.
2nd edition, Mc GrawHill, NewYork.

67. MISHRA, R.B. (1975)
Optimum production lot size model for a system with deteriorating inventory.
Int. Jr. Prod. Res., 13, 495-505.

68. MITRA, A. COX, J.P. and JESSES, R.R. (1984)
A note on detemining order quantities with a linear trend in demand
J. Opl. Res. Soc., 31, 615-620.

69. MURDESHWAR, T.M and SATHE, Y.S. (1985)
Some aspects of lot size models with two levels of storages.
Opsearch, 22 (4), 255-262.

70. MUKHERJEE, S.P. (1987)
Optimum ordering interval for time varying decay rate .of inventory.
Opsearch, 24 (1), 19-24.

71. MURDESHWAR, T.M. (1988)
Inventory replenishment policy for linearly increasing demand considering shortages—an optimal solution.
J. Opl. Res. Soc., 39, 687-692.

72. MANDAL, B.N. and PHAUJDAR, S. (1989)
A note on an inventory model with stock-dependent consumption rate.
Opsearch, 26 (1), 43-46.

73. MANDAL, B.N. and PHAUJDAR (1989)
An inventory model for deteriorating items and stock-dependent consump-

tion rate.
J. Opl. Res. Soc., 40 (5), 483-488.

74. MOINZADEH, KAMRAN and LEE, HAUL (1989)
Approximate order quantities and re-order points for inventory systems where orders arrive in two shipments.
Operations Research, 17 (2), 277-287.

75. NADDOR, E (1966)
Inventory Systems.
John Wiley and Sons, New York.

76. NAHMIAS, S. and PIEKSKALLA, W.P. (1973)
Optimal ordering policies for a product that perishes in two periods subject to stochastic demand.
Nav. Res. Log. Quart., 22, 207-229.

77. NAHMIAS, S. and PIEKSKALLA, W.P. (1975)
Optimal ordering policies for perishable inventory-I, proceeding of 20th International meeting.
The Inst. of Mgmt. Science, Vol. II, 485-493.

78. NAHMIAS, S. (1975),
Optimal ordering policies for perishable inventory II.
Operations Res., 23, 735-749.

79. NAHMIAS, S. (1975)
On ordering perishable inventory under Erlang demand.
Nav. Res. Log. Quart., 22, 415-425.

80. PARSONS, J.A. (1966)
Multi product lot size determination when certain restrictions are active.
J. Engng., 17, 360-365.

81. PHILIP, G.C. (1974)
A generalized EOQ model in items with Weibull distribution deterioration.
AIIE Trans., 6, 159-162.

82. PHELPS, R.I. (1980)
Optimal inventory rule for a linear trend in demand with a constant

replenishment period.
J. Opl. Res. Soc., 31, 439-442.

83. PLOSSL, GEORGE (1986)
Inventory Management.
Printice Hall of India.

84. RICHARDS, F.R. (1975)
Comments on the distribution of inventory position in a continuous review (s, S) systems with constant lead times.
Opns. Res., 27, 717-729.

85. RITCHIE, E. (1984)
The EOQ for linear increasing demand : a simple optimal solution.
J. Opl. Res. Soc., 35, 949-952.

86. RAVI CHANDRAN, N. (1984)
A note on (s, S) inventory policy.
AIIE Trans., 16, 387-390.

87. STARR and MILL (1962)
Inventory Control : Theory and Practice.
Printice Hall, Englewood, Cliffs.

88. SILVER, E.A. and MEAL, H.C. (1969)
A simple modification of the EOQ for the case of a varying demand rate.
Prod. Invent. Mgmt., 10, 52-65.

89. SILVER, E.A. (1970)
Some ideas related to the inventory control of items having erratic demands pattern.
CORSJ, 8, 88-100.

90. SILVER, E.A. and MEAL, H.C. (1973)
A heuristic for selecting lot size quantities for the case of a deterministic time varying demand rate and discrete opportunities for replenishment.
Prod. Invent. Mgmt., 14, 64-74.

91. SHAH, Y.K. and. JAISWAL, M.C. (1977)
An order-level inventory model for a system with constant rate of deterioration.
Opsearch, 14 (3), 174-184.

92. SHAH, Y.K. and JAISWAL, M.C. (1977)
A periodic review inventory model for items that deteriorate continuously in time.
Inter. Jour. Prod. Res.

93. SHAH, Y.K. and JAISWAL, M.C. (1977)
A probabilistic scheduling inventory model for deteriorating items.

94. SHAH, Y.K. (1977)
An order level lot size inventory model for deteriorating items.
AIIE Trans., 9 (1), 108-112.

95. SILVER, E.A. (1979)
A simple inventory replenishment decision rule for a linear trend in demand.
J. Opl. Res. Soc., 71-75.

96. SACHIN, I. (1979)
On the stationary analysis of continuous review (s, S) systems with constant lead times.
Opns. Res., 27, 717-727..

97. SHARMA, K.V.S. (1983)
A deterministic inventory model with two levels of storage and an optimum release rule.
Opsearch, 20 (3), 175-180.

98. SACHAN, R.S. (1984)
On (T, S_i) policy inventory model for deteriorating items with time propotional demand.
J. Opl. Res. Soc., 1013-1019.

99. SHARMA, K.V.S. (1984)
On a K-release rule for an inventory system with two ware houses.
Opsearch, 21 (1), 38-40.

100. SILVER, E.A. and PETERSON, R. (1985)
Decision systems in inventory management and production planning.
Ind. Edition, Wiley, NewYork.

101. SCHULTZ, CARL. R. (1987)
Forecasting and inventory control for sporadic demand under periodic review.
J. Opl. Res. Soc., 38 (5), 453-458.

102. SATIR, AHMET and CENGIZDILEK (1987)
Medicinal inventory control in a University health centre.
J. Opl. Res. Soc., 38 (5), 387-395.

103. TERSINI, R.J. (1982)
Principals of Inventory and Material Management.
Ind. Edition, North Holland, New York.

104. TAYLOR, S.G. and BRADLEY, C.E. (1985)
Optimal ordering strategies for announced price increase.
Operations Research, 312-325.

105. VANZYL, G.G.J. (1964)
Inventory control for perishable commodities, unpublished doctoral dissertation.
Uni. of North Carolina, Chapel Hill, N.C.

106. WAGNER, H.M. and WHITIN, T.M. (1958)
Dynamic version of the economic lot size model.
Management Science., 5, 89-96.

107. WARD, J.B. (1978)
Determining reorder points when demand is lumpy.
Management Science, 24, 623-632.

108. WILLIAMS, T.M. (1982)
Reorder levels for lumpy demand.
J. Opl. Res. Soc., 185-189.

109. WARD, S.C. and CHAPMAN, C.B. (1987)
Inventory control and trade credit—a reply to Daellenbach.

J. Opl. Res. Soc., 1081-1084.

110. YANASSE, HORACIOHIDEK (1990)
EOQ Systems : The case of an increase in purchase cost.
J. Opl. Res. Soc., 41 (7), 633-637.

111. ZANGWILL, W.I. (1966)
A deterministic multi-period production scheduling model with back logging.
Management Science, 13, 105-119.

112. ZIVAZLIAN, B. (1974)
A continuous review (s, S) inventory system with arbitrary distribution between unit demand.
Opns. Res., 22, 65-71.